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TOPICAL REVIEW

Critical phenomena and universal dynamics in one-dimensional driven diffusive systems with two species of particles

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Abstract

Recent work on stochastic interacting particle systems with two particle species (or single-species systems with kinematic constraints) has demonstrated the existence of spontaneous symmetry breaking, long-range order and phase coexistence in nonequilibrium steady states, even if translational invariance is not broken by defects or open boundaries. If both particle species are conserved, the temporal behaviour is largely unexplored, but first results of current work on the transition from the microscopic to the macroscopic scale yield exact coupled nonlinear hydrodynamic equations and indicate the emergence of novel types of shock waves which are collective excitations stabilized by the flow of microscopic fluctuations. We review the basic stationary and dynamic properties of these systems, highlighting the role of conservation laws and kinetic constraints for the hydrodynamic behaviour, the microscopic origin of domain wall (shock) stability and the coarsening dynamics of domains during phase separation.

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1. Introduction

1.1. Why we are interested in driven diffusive systems

The investigation of interacting particle systems far from equilibrium has shown that one-dimensional driven diffusive systems with short-range interactions exhibit a remarkably rich variety of critical phenomena. Unlike in thermal equilibrium one observes spontaneous symmetry breaking, long-range order and phase coexistence in the steady state if the system evolves under certain microscopic kinetic constraints or has more than one conservation law. A large body of work has been devoted to microscopic stochastic lattice models for driven

diffusive systems where classical interacting particles move under the action of a random force preferentially in one direction[1].

The ongoing interest in these systems is for many reasons. The most obvious one is derived from a fundamental task of statistical mechanics, namely the desire to understand the emergence of macroscopic collective properties from microscopic interactions, with a view on general features such as interaction range (short- or long-range), kinetic constraints or the presence of conservation laws. It has turned out that parts of that program can be carried out to a very satisfactory degree in the simplest case of driven diffusive systems of identical conserved particles with hard-core interaction. Despite their simplicity, these systems exhibit a rich and rather non-trivial dynamical and stationary behaviour. For an exactly solvable paradigmatic model, the asymmetric simple exclusion process (ASEP, see below), not only the macroscopic nonlinear hydrodynamics have been derived rigorously [1, 3] but also detailed information about *universal* phenomena, including shock diffusion [2], the microscopic origin of the stability of shocks [4] and the dynamical structure function [5] could be obtained in the past decade. It is then natural to ask what to expect in the presence of more than one conservation law, i.e., in systems with several distinct species of particles.

Secondly, in the absence of a general framework for studying nonequilibrium systems (analogous to the usual principles of equilibrium statistical mechanics), one needs to understand coarse-grained dynamical properties not only for their own sake, but also in order to predict what stationary states these systems evolve into. For one-species systems with open systems this has led to a theory of boundary-induced phase transitions which provides a general framework for a quantitative description of the steady-state selection in driven diffusive systems which are in contact with particle reservoirs at their boundary. Unlike in equilibrium, boundary conditions determine the bulk behaviour of driven diffusive systems in a decisive fashion which can be captured in terms of an extremal principle for the current [6, 7]. The resulting phase diagram for the nonequilibrium steady state is determined by the interplay of localized excitations and shocks. Again it is natural to ask for principles of steady-state selection and the resulting phase diagram in systems with many species of particles which are characterized by a conserved current for each particle species.

A third motivation for studying these systems stems from numerical evidence which shows that addressing these questions not only leads to an encyclopedic accumulation of knowledge. Rather it was found that there is exciting new physics in systems with more than one species of particles, including spontaneous symmetry breaking and phase separation phenomena, even in translation-invariant systems [8, 9] without the beneficial 'assistance' of open boundaries or static defects [10] in facilitating phase transitions. It also emerged that similar phenomena may occur in single-species systems with one conservation law, provided there are kinetic constraints determined by a zero-rate condition somewhat analogous to the zero-temperature condition for long-range order in equilibrium systems with short-range interactions. Neither the hydrodynamic behaviour of systems with more than one conservation law nor the microscopic conditions for the occurrence of critical phenomena are well understood.

These are some—and by far not all—of the fundamental reasons to investigate translation-invariant driven diffusive systems and to clarify the role of conservation laws and kinetic constraints for their dynamical and steady-state properties. Other important issues include the calculation of exact stationary distributions and large deviation functions, nonequilibrium

[1] Strictly speaking one should call these models mesoscopic as interactions on scales below the particle size are replaced by effective interactions. However, the notion microscopic has become standard and will be used here. It is justified in relation to a truly macroscopic description where particle positions are replaced by coarse-grained density fields.

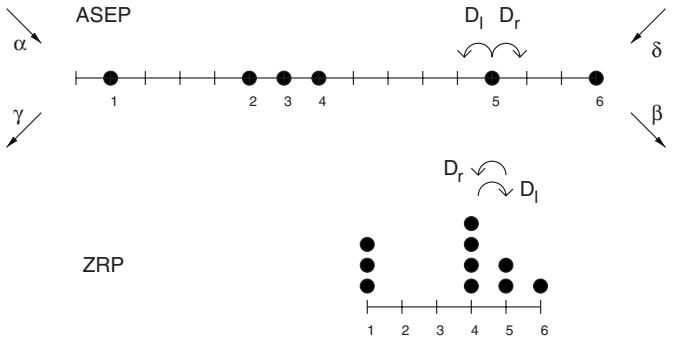

Figure 1. Asymmetric simple exclusion process and related zero-range process. In the ASEP, particles on a lattice hop with rates $D_{r,l}$ to the right and left respectively, provided the target site is empty. In the case of open boundaries they are created or annihilated with rates $\alpha, \beta, \gamma, \delta$ at the boundary sites $1, L$ as indicated in the figure. Reflecting boundaries correspond to $\alpha = \beta = \gamma = \delta = 0$, in the case of periodic boundaries particles may hop between sites $L, 1$. In the associated zero-range process consecutive particles correspond to sites, the length of empty intervals between them to occupation numbers. Hopping of particle k from a given site in the ASEP corresponds to hopping between sites $k-1, k$ in the ZRP.

Yang–Lee theory and phase transitions in systems with absorbing states. Some of these topics are discussed in a complementary review by Evans [11]. We also refer to [12–15] where closely related questions in nonconservative systems and higher dimensions are treated and to [16] and references therein for phase transitions in systems with continuous state space. Here we focus on the issues of hydrodynamic behaviour, the microscopic origin of domain wall (shock) stability, spontaneous symmetry breaking and coarsening of domains in one-dimensional conservative lattice systems, with particular emphasis on models with two conservation laws and on exclusion processes with one conservation law insofar that they are relevant in the context of the nonequilibrium bulk phase transitions that we review here.

1.2. Basic models

The asymmetric simple exclusion process [3, 4] has become a paradigmatic example for a driven diffusive system and has begun to attain a status in the study of nonequilibrium systems somewhat similar to the role the Ising model plays in equilibrium statistical mechanics. In this stochastic lattice gas model each lattice site is occupied by at most one particle. Particles hop randomly in continuous time to the right neighbouring site with rate D_r and to the left with rate D_ℓ respectively, provided the target site is empty. Otherwise the attempted move is rejected. We present this hopping rule as follows:

$$\begin{aligned} A0 &\to 0A \quad \text{with rate} \quad D_r \\ 0A &\to A0 \quad \text{with rate} \quad D_l. \end{aligned} \tag{1.1}$$

Hopping attempts take place independently with an exponential waiting time distribution with mean $D_r + D_\ell$ (figure 1).

These simple rules specify completely the stochastic bulk dynamics of the system[2]. For a finite lattice with L sites one has to specify boundary conditions. Most commonly studied are periodic boundary conditions, reflecting boundaries (hopping confined to a box), and open boundary conditions where particles may enter and exit the lattice at the boundary sites 1 and L

[2] For a mathematically precise definition of the process, see [3, 17].

respectively with rates α, β, γ, δ (see figure 1). By choosing $\alpha = \rho_1 D_r \lambda_1$, $\gamma = (1-\rho_1) D_l \lambda_1$ as left boundary rates and $\beta = (1-\rho_2) D_r \lambda_2$, $\delta = \rho_2 D_l \lambda_2$ the open system may be interpreted as being connected to particle reservoirs with constant density ρ_1 at the left (auxiliary) boundary site 0 and density ρ_2 at the right (auxiliary) boundary site $L+1$ respectively. The parameters $\lambda_{1,2}$ are introduced to describe a hopping mechanism between the reservoirs and the chain which may differ from the hopping inside the chain. From a physics point of view this would correspond to activation energies for entering (leaving) the system which are different from those for hopping in the bulk.

This is the simplest model that incorporates the basic features of a driven diffusive system with short-range interactions. The short-range interaction is taken care of by the hard-core exclusion constraint. The randomness of the hopping events models diffusive motion of the free particles outside the interaction range. The hopping asymmetry corresponds to a driving force that leads to a biased average motion and hence to a macroscopic particle current even in the stationary state of the system, except in the case of reflecting boundary conditions, where the system reaches an equilibrium state [17, 18]. Throughout this paper we assume a bias in positive lattice direction.

Note that each lattice site can be in two states, either occupied or empty, and hence the system can be described in terms of occupation numbers $n_k = 0, 1$. Implicit in this description is the absence of any internal degree of freedom that particles may possess. Hence all particles are indistinguishable. The number of particles is conserved in the bulk, but not at the boundaries in the case of the open system. The single bulk conservation law gives rise to a current via the lattice continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_k = j_{k-1} - j_k \tag{1.2}$$

for the expected density $\rho_k = \langle n_k \rangle$, averaged over realizations of the stochastic time evolution and also averaged over different initial distributions. For the ASEP the current follows straightforwardly from the definition,

$$j_k = D_r \langle n_k (1 - n_{k+1}) \rangle - D_\ell \langle (1 - n_k) n_{k+1} \rangle. \tag{1.3}$$

For periodic boundary conditions there is a family of stationary distributions which are Bernoulli product measures with density ρ, i.e., at each given site the probability of finding a particle is given by ρ, independent of the occupation of other sites. According to (1.3) the stationary current

$$j = (D_r - D_\ell)\rho(1-\rho) \tag{1.4}$$

is a nonlinear function of the density with a single maximum at $\rho = \frac{1}{2}$.

If the ASEP is confined to a box (corresponding to 'open boundaries' with $\alpha = \beta = \gamma = \delta = 0$) the system evolves into an equilibrium state where essentially all N particles form a cluster of macroscopic size $\approx N$ with density $\rho \approx 1$ and the current vanishes. The density profile has a non-trivial form (deviating significantly from 0 or 1 respectively) only at the left edge of the cluster, denoted below as 'shock', 'domain wall' or 'interface' respectively, depending on context. The width of this interface is finite on the lattice scale, i.e., microscopic. In the grand-canonical ensemble there are no correlations, but the density profile has the form of a hyperbolic tangent [17]. The canonical distribution is more complicated, with correlations within the interface region, but has a similar density profile [18].

The open ASEP has a intriguing phase diagram with a nonequilibrium first-order transition at $\rho_1 = 1 - \rho_2$ between a low-density phase with bulk density $\rho = \rho_1$ to a high-density phase with bulk density $\rho = \rho_2$. There are nonequilibrium second-order transitions from both phases to a maximal current phase with $\rho = \frac{1}{2}$, irrespective of the boundary densities in the square

defined by $\rho_1 > \frac{1}{2}$, $\rho_2 < \frac{1}{2}$. The density profiles are non-trivial in all phases [19, 20]. At the first-order transition line one has phase coexistence with domains of densities $\rho_{1,2}$, separated by a microscopically sharp domain wall (shock), the position of which performs a random walk over the whole lattice. The exact solution of the stationary density profiles and a theory of boundary-induced phase transitions which provides a microscopically oriented derivation of the phase diagram is reviewed in detail in [4]. The bulk densities as a function of boundary densities were obtained by Liggett [21] using probabilistic methods. More generally, the theory of boundary-induced phase transition which takes into account the flow of fluctuations has revealed that the bulk density that simple open driven diffusive systems select can be obtained from the extremal principle [7]

$$\begin{aligned} j &= \max_{\rho\in[\rho_2,\rho_2]} j(\rho) \qquad \text{for} \quad \rho_1 > \rho_2 \\ j &= \min_{\rho\in[\rho_1,\rho_2]} j(\rho) \qquad \text{for} \quad \rho_1 < \rho_2 \end{aligned} \tag{1.5}$$

involving the stationary current–density relation that can be obtained, e.g., from measurements or exact calculations in periodic systems where the density is conserved. The boundary densities $\rho_{1,2}$ entering (1.5) are non-universal functions of the rates at which particles enter and leave the system. For $\rho_2 = 0$ the extremal principle reduces to a maximal current criterion which was first proposed by Krug [6] on a phenomenological basis.

One may relax the exclusion constraint to allow for up to m particles on each lattice site. This gives rise to the partial exclusion process [22, 23] with $m + 1$ states per site. Also particle systems with nonconserved internal degrees of freedom such as velocities in traffic flow models [24, 25] have more than one possible state per site, but still obey a single continuity equation of the form (1.2). The form (1.3) of the current, however, strongly depends on the microscopic hopping rules. An important model without exclusion is obtained by the following simple mapping of the ASEP: since the order of particles is conserved in the ASEP one may regard particles as sites of a new lattice gas system and the number of vacancies n_j between particles j, $j + 1$ as occupation numbers on site j. This gives rise to a special case of the zero-range process (ZRP) [26]. The particle hopping rates in the ASEP turn into the rates of decreasing the number of particles by one unit in the ZRP, with hopping to the right in the ASEP corresponding to hopping to the left in ZRP and vice versa (figure 1).

The general homogeneous ZRP allows for hopping of a particle from site j with a rate w_n that depends only on the unrestricted occupation number n_j. Here we shall consider only nearest neighbour hopping (figure 2) with asymmetry factors p, q to the right and left respectively. Below a critical density (which may be infinite, see section 4), the periodic and infinite system has a family of stationary distributions which are product measures where the probability of finding n particles on a given site is given by [26, 27]

$$p_0 = \frac{1}{Z} \qquad p_n = \frac{1}{Z} z^n \prod_{i=1}^{n} 1/w_i \tag{1.6}$$

with the 'fugacity' z fixing the mean particle density and the one-site nonequilibrium analogue

$$Z = \sum_{n=0}^{\infty} z^n \prod_{i=1}^{n} 1/w_i \tag{1.7}$$

of the partition function. The density as a function of z is then given by

$$\rho = z \frac{\mathrm{d}}{\mathrm{d}z} \ln Z. \tag{1.8}$$

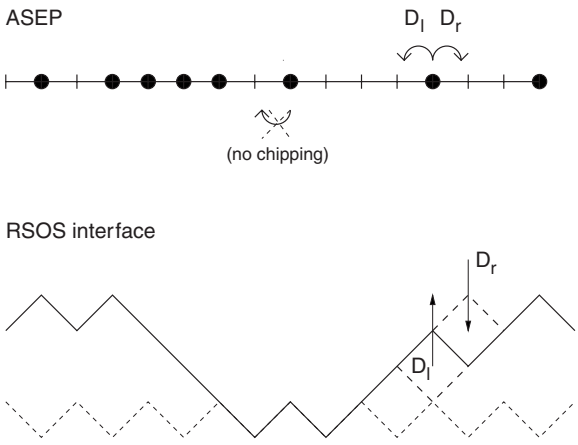

Figure 2. Mapping of the ASEP to a restricted solid-on-solid (RSOS) growth model. An empty (occupied) site corresponds to a slope +1 (−1) in the associated interface. Hopping to the right (left) thus is equivalent to a deposition (evaporation) of an atom. The broken line at the bottom of the interface marks a completed layer. In the absence of chipping the interface cannot shrink below a completed layer which yields a minimal height model. For an anchored interface the minimal height of the interface equals the height of the topmost completed layer. In two-species models the definitions are analogous, but with allowed local interface slopes $0, \pm 1$.

According to its definition via the lattice continuity equation for the ZRP the stationary particle current is given by

$$j = (p-q)\sum_{n=1}^{\infty} p_n w_n = (p-q)z. \tag{1.9}$$

The density dependence of j in the stationary state can be obtained by inverting the relation $\rho(z)$ which is a monotonically increasing function of ρ [8]. Note that the radius of convergence depends on the choice of rates w_i (see section 4).

In order to describe a system with two different conserved species A, B of identical particles (or alternatively, tagged particles or particles with two internal states which do not affect its dynamics) one needs a model where each lattice site can be found in at least three different states: empty or occupied by either an A-particle or a B-particle. The most simple extension of the exclusion process that accounts for the possibility of two particle species may hence be described by the six hopping rates

$$\begin{array}{lll} A0 \to 0A & \text{with rate} & D_{A0} \\ 0A \to A0 & \text{with rate} & D_{0A} \\ B0 \to 0B & \text{with rate} & D_{B0} \\ 0B \to B0 & \text{with rate} & D_{0B} \\ AB \to BA & \text{with rate} & D_{AB} \\ BA \to AB & \text{with rate} & D_{BA}. \end{array} \tag{1.10}$$

There is no established name for this generic process and we shall refer to it as two-species ASEP. Associated with the two conservation laws there are two currents defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_k^A = j_{k-1}^A - j_k^A \tag{1.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_k^B = j_{k-1}^B - j_k^B. \tag{1.12}$$

Note that in general j^A and j^B depend on both occupation numbers n_k^A, n_k^B respectively. Hence one has two coupled lattice continuity equations. The stationary distribution of this process and hence the current–density relation is known only on certain parameter manifolds, see below.

The natural order parameter that describes the macroscopic state of the system is the particle density of each species. Hence for each conservation law there is an associated order parameter. Note, however, that one may have conservation laws that are only indirectly related to particle densities which by themselves may not be conserved. For example, in a reaction–diffusion system $A + B \to 0$ [28] where A- and B-particles annihilate upon encounter (i.e., react into an inert reaction product), the difference $s = n^A - n^B$ still gives rise to a single conservation law with an associated current, even though n^A and n^B are not individually conserved. By interpreting A-particles (B-particles) as carriers of a positive (negative) electrical charge, one could speak in this case of charge conservation. In the two-species catalytic reaction $A + B \to B + B$ [29] the total density $n = n^A + n^B$ is conserved. Analogously, one may also consider models with two conservation laws, but more than three states per site. Examples include two-lane models [30, 31] or bricklayer models [32].

There is no answer to the question to which extent or under which circumstances the existence of internal degrees of freedom matters for the macroscopic properties of driven diffusive systems. Since, however, it is now clear that the number of conservation laws is important, we shall categorize the models in the following according to this property.

1.3. Closely related models, not covered in this review

Equivalence to two-dimensional equilibrium systems. The time-dependent one-dimensional stochastic processes discussed above are equivalent to two-dimensional equilibrium systems, defined by some vertex model [33]. The time evolution is encoded in the transfer matrix of the two-dimensional (2D) model, e.g., the discrete-time ASEP with a sublattice parallel update corresponds to the six-vertex model [34–37]. For three-state models the construction is entirely analogous and leads to higher vertex models. Recently also integrable vertex models with an unlimited number of states have been investigated [38]. They correspond to the zero-range representation of the ASEP. The stationary distribution of a one-dimensional process with L sites corresponds to the equilibrium state of the associated two-dimensional model, defined on a strip of dimension $L \times \infty$. Processes defined in continuous time are derived from the transfer matrix of the vertex model in the same way as one obtains quantum spin-chain Hamiltonians [4]. Hence the Markov generator of the stochastic time evolution is equivalent to some (usually non-Hermitian) one-dimensional quantum Hamiltonian. This opens the tool box of condensed-matter physics for the study of stochastic dynamics. The ASEP (1.1), the two-species ASEP (1.10) on a certain parameter manifold [129] and also various single-species [39, 40] and two-species reaction–diffusion models [29, 39, 41–43] correspond to integrable models, for which Bethe ansatz and related methods yield exact results on the dynamics of the system.

We note that by considering the stationary states of one-dimensional nonequilibrium systems as equilibrium distributions of two-dimensional systems the occurrence of long-range order and phase separation becomes somewhat less mysterious. However, the important question of how these phenomena emerge from the microscopic laws of interaction cannot be answered by this formal equivalence. We mention in passing that in another mapping the steady-state distribution describes the equilibrium properties of a directed polymer in a 2D random energy landscape [44].

Higher dimensional nonequilibrium models. The extension of the models discussed above to higher dimensions is technically rather straightforward and obviously of importance. For suitably chosen initial distribution some features of shocks may be present in higher dimensional driven diffusive systems, but many of the physical properties of the one-dimensional systems discussed below are expected to change dramatically not only because of the upper critical dimension $d_c = 2$ for diffusion (which makes mean-field behaviour more likely to describe the systems), but also because of the absence of blocking effects due to hard-core exclusion. Moreover, there may be phase transitions in the unbiased equilibrium counterparts of the model which lead to new phenomena in the biased case. Two-dimensional driven diffusive systems are reviewed in detail in [12].

Many conservation laws. A natural, but for the scope of this review too far-reaching, question is the behaviour of particle systems with many conservation laws. Not surprisingly, no general picture has emerged yet. An interesting generalization of the ASEP comprises lattice systems with particles covering more than one lattice site, but moving only by one site in each infinitesimal time step [45–49]. Remarkably, these models are also integrable, including polydisperse models with particles of different sizes (and hence as many conservation laws).

Another class of models with many conservation laws arises from assigning to each particle its own intrinsic hopping rate. In the zero-range mapping one thus obtains a process with site-dependent quenched random hopping rates. For rates drawn from some distribution the hydrodynamic behaviour has been studied in [50–52]. For certain distributions the system with asymmetric hopping rates undergoes at some critical density a transition to a platoon state, where fast particles are trailing a slow one reminiscent of traffic flow. This transition is a classical analogue of Bose–Einstein condensation [53, 54] and is quite analogous to the condensation transition in ordered systems to be discussed in detail below. A similar model with passing of particles has also been studied [55, 56]. Remarkably, the phase diagram of the open system both with [57] or without passing [58] has a structure similar to that of the usual one-species exclusion process [19, 20] which can be explained by the theory of boundary-induced phase transitions for systems with one conservation law [7, 59].

2. Applications

The motivation we gave for studying the systems reviewed here addressed general questions of nonequilibrium statistical mechanics, with little reference to actual realizations where such processes might play a role. Applications are actually numerous and include not only quasi-one-dimensional settings (e.g. molecular diffusion in nanoporous materials such as zeolites [60–65], single-file diffusion of mesoscopic colloidal particles [66] or ionic conduction in narrow channels [67–69]) but also—through various mappings—two- and three-dimensional systems. Of course, the basic models (1.1) and (1.10) can serve only as very crude approximations for any real complex system. However, the universality of critical phenomena (dynamical and static), of diffusion, of the emergence of shocks and of coarsening allows for the study of fundamental properties of real systems in terms of simple toy models. It is not the purpose of this paper to review such applications in any detail, but some significant results are summarized.

Tracer diffusion. The simplest way of obtaining a system with two conservation laws consists in considering tagged particles in the usual ASEP. Tagged particles (=particles of type B) have

the same physical properties as the usual particles, except that they carry a marker which allows for their identification, but does not affect the dynamics. Thus one gets the two-species ASEP (1.4) with

$$D_{B0} = D_{A0} \qquad D_{0B} = D_{0A} \qquad D_{AB} = D_{BA} = 0. \tag{2.1}$$

In the unbiased case $D_{A0} = D_{0A}$ a single tracer particle in a stationary system of density ρ is predicted to perform anomalous diffusion with a mean-square displacement $\langle X^2(t)\rangle \propto (1-\rho)/\rho\sqrt{t}$ [70–72]. Recently, this was confirmed experimentally in the investigation of tracer diffusion in zeolites [61] using pulsed field gradient NMR [73] and in the study of single-file diffusion of colloidal particles [66].

Also a driven tracer particle in an environment of unbiased A-particles behaves subdiffusively with a square-root power law for the mean-square displacement [74]. In the fully driven case (2.1) the situation is more complex. When averaging over random initial states of the system according to the weights given by the stationary distribution, the mean-square displacement was proved to grow linearly in time with a diffusion coefficient $D = (D_r - D_\ell)(1-\rho)$ [75]. On the other hand, for fixed initial states (averaging only over realizations of the process) the variance is expected to grow subdiffusively with power $t^{2/3}$ [76, 77]. In a finite system with periodic boundaries the variance in the number of hops made in the totally asymmetric process ($D_{0A} = 0$) has been calculated exactly in the infinite-time limit [78, 79] and been found to decrease asymptotically $\propto 1/\sqrt{L}$ in system size. This is to be expected from dynamical scaling with the well-known dynamical exponent $z = \frac{3}{2}$ of the asymmetric exclusion process [80, 81].

Shock tracking. The ASEP exhibits shocks which on a macroscopic level appear as stable moving discontinuities in the density profile. It is of great interest to understand the microscopic structure of shocks, i.e., the density profile and correlations on the microscopic lattice scale which for a real system is the analogue of intermolecular distances. The fundamental question is whether the density changes quickly over molecular length scales or much slower (but still abruptly on macroscopic scales). Trying to answer this question leads to the problem of defining a microscopic shock position in a given realization of the process. This can be accomplished by introducing a second-class particle, i.e., a particle that moves w.r.t. vacancies like an ordinary (first-class) particle, but behaves like a vacancy w.r.t. to ordinary particles [82]. This leads to rates (1.10) with

$$D_{B0} = D_{A0} \qquad D_{0B} = D_{0A} \qquad D_{AB} = D_{A0} \qquad D_{BA} = D_{0A}. \tag{2.2}$$

By studying the motion of a single second-class particle one finds the mean shock velocity

$$v_s = \frac{j_1 - j_2}{\rho_1 - \rho_2} \tag{2.3}$$

for a shock jumping from a density ρ_1 to ρ_2 with stationary current $j_{1,2}$ in each domain. Expression (2.3) may be deduced directly from mass conservation. For fixed initial states the variance of the shock position is subdiffusive with power law $t^{1/3}$ [76], while with averaging over random initial states with stationary weights at different densities $\rho_{1,2}$ to the right and left of the shock (i.e. starting the system from a shock measure) one finds ordinary diffusion. The diffusion coefficient was conjectured [83] and subsequently proved [84] to be given by

$$D_s = \frac{1}{2}\frac{j_1 + j_2}{\rho_1 - \rho_2}. \tag{2.4}$$

More detailed information about the microscopic structure of the shock has been proved by a variety of methods, for a review see [2] and for more recent work [85–87] and references

therein. It has been established that a shock is truly microscopic in the sense that a rapid increase in the density of particles occurs on the lattice scale. Loosely speaking, one may say that the shock performs a random walk with drift velocity (2.3) and diffusion coefficient (2.4). Some details concerning the structure of a shock are reviewed below.

In a region of smooth variation of the density the second-class particle allows for tracking localized perturbations of the density [5, 88]. The mean velocity of the second-class particle is given by the collective velocity

$$v_c = \frac{\partial}{\partial \rho} j(\rho) \tag{2.5}$$

of a density perturbation. Its mean-square displacement grows superdiffusively with power law

$$\langle X^2(t) \rangle - \langle X(t) \rangle^2 \propto t^{4/3} \tag{2.6}$$

where the divergent effective diffusion coefficient Δ crosses over in a finite system to [90]

$$\Delta \sim L^{\frac{1}{2}} \tag{2.7}$$

which is expected from dynamical scaling with dynamical exponent $z = \frac{3}{2}$. As a toy model for econophysics the position of the second-class particle marks the price of an asset on the price axis in a limit order market, the first-class particles represent bid and ask prices [89]. The rigorous result (2.6) corresponds to a Hurst exponent $H = \frac{2}{3}$ for fluctuations which compares well with the empirically observed value $H \approx 0.6$ for intermediate time ranges. Introducing annihilation and creation of particles (cancellation and renewal of orders) leads to the Gaussian value $H = \frac{1}{2}$ after some crossover time, also in agreement with empirical findings.

Traffic flow. The occurrence and microscopic nature of shocks in the ASEP are reminiscent of traffic jams in vehicular traffic. Indeed, traffic flow may be regarded as a driven diffusive system [24, 25], albeit with a nonconserved internal degree of freedom, namely the speed of cars which is dynamically determined by the competition of the desire to move at an optimal high speed and the necessity to keep a velocity-dependent minimal safety headway to the next car. At low densities the mean distance between cars is larger than the required safety headway and essentially all cars move at their optimal speed, with some fluctuations. At high densities cars have a mean distance below the safety headway corresponding to the optimal speed which leads to a slowing down of the traffic. As a result the mean current as a function of density ρ (known as fundamental diagram in the traffic literature) has a maximum like the exclusion process, albeit with a much broader distribution of the current. In the first measurement of traffic flow in 1935 by Greenshields [91] the measured mean flow of cars was approximated by expression (1.4) $j \propto \rho(1 - \rho)$ that the ASEP yields. More sophisticated models that provide a much better description of real traffic data have been developed in the past decade, starting with the Nagel–Schreckenberg model [92] which contains the ASEP as a simple limiting case. Yet some fundamental features of the ASEP survive in the more complicated Nagel–Schreckenberg model. The theory of boundary-induced phase transitions developed in [7, 59] which predicts the stationary phase diagram of a single-species system with open boundaries in terms of the extremal principle (1.5) explains quantitatively the phase diagram of the Nagel–Schreckenberg model in terms of effective boundary densities and is also consistent with measurements of real highway traffic [93].

A description of traffic flow with a lattice model with one conservation law corresponds to modelling cars which all have the same intrinsic optimal speed—a rather crude approximation if describing mixed traffic of cars and trucks is envisaged. This naturally leads as a next

approximation step to a two-species description with 'fast' and 'slow' particles respectively. A model with two conservation laws arises also from the study of two-way traffic flow with interaction between lanes, but no exchange of particles in each lane. Taking the exclusion process as the simplest possible model one arrives at the model (1.10) with suitably chosen rates [94–96], see below. A mixture of cars with individual intrinsic speeds leads to the disordered hopping models mentioned above.

Biophysics. The ASEP with open boundaries was first developed as a simple model for describing the kinetics of protein synthesis [97, 98]. Here particles are ribosomes moving along the codons of a messenger RNA. Each codon corresponds to a specific amino acid which the ribosome uses to assemble a protein. When such a step is completed the ribosome moves on to the next codon and continues with the addition of the next amino acid to the growing protein molecule. The injection of particles at one end marks the initialization of the process, the absorption at the other boundary describes the release of the ribosome. The shock known from the exclusion process corresponds to a traffic jam of ribosomes which explains an experimentally observed slowing down of the ribosomes as they approach the terminal point of the m-RNA where they are released after completion of the protein synthesis [48, 49, 99, 100].

Very recent work has shown that in another biological setting exclusion particles may describe molecular motors such as kinesins moving along microtubuli or actin filaments in a cell [101, 102]. Due to attachment and detachment during the motion a description with nonconservative dynamics where particles are annihilated and created also in the bulk with a small rate is required. This leads to the model of [89] with open boundaries which yields interesting new phenomena [103]. Oppositely moving molecular motors give rise to a two-species exclusion model, in analogy to two-way traffic flow with interaction between lanes.

A two-species exclusion process has also been introduced to describe the motion of ants along ant trails [104]. While crawling along a trail, the ants—modelled by A-particles hopping along a lattice—produce pheromones (B-particles) which serve as a marker of the traversed path for other ants which again produce pheromones for subsequent ants. This is necessary to stabilize the trail as the pheromones evaporate after some time. The pheromones are modelled as an immobile particle species which is deposited when a hopping event has taken place and which disappears with some evaporation rate. The analogy of the flow of ants to traffic flow has been pointed out in [105] who measured the flow rate versus the density of ants, i.e., the current–density relation. The numerical results obtained from the two-species ant trail model yield qualitatively similar results [104]. Essentially, the same model (with different parameter values and update rules) has been introduced as a 'bus route' model where one observes bunching of particles (='buses') as they travel along the lattice ('bus stops') and pick up passengers [106]. Bunching of real buses appears to occur on services which do not run according to fixed schedules, but which stop according to demand.

Polymer dynamics. In polymer networks such as rubber gum or gels, in polymer melts or in dense solutions of macromolecules such as DNA different polymer strands form a complicated topological structure of entanglements somewhat reminiscent of a large portion of spaghetti. The entanglements severely restrict the dynamical degrees of freedom of the polymer chains. In the framework of the celebrated reptation theory [107] developed by Doi, Edwards and de Gennes [108, 109] the motion of an individual polymer is viewed as being confined by a hypothetical tube which models the collective effect of all entanglements of the neighbouring polymer chains. In an uncrosslinked melt or solution the tube is open at both ends, since at the end points the motion of polymer segments transverse to its own contour is not restricted

by topological constraints. This picture results in a snake-like one-dimensional effective dynamics of polymer segments along the tube, with extra orientational degrees of freedom only at its ends.

In a lattice model of Rubinstein [110] the reptation dynamics is modelled by the symmetric exclusion process (1.1) with open boundaries which describe the extra end point degrees of freedom. With this model exact results for the relaxation of the contour and contour length fluctuations have been obtained. Recent experiments on the dynamics of single entangled DNA-molecules in dense solution confirm the findings [111, 112].

Duke [113] extended the model to allow for tracking the spatial orientation of the tube rather than only its length. This was done in order to introduce a reference axis for describing gel electrophoresis, i.e., the separation of polymer fragments according to their length L. By applying an electric field of strength E (the direction of which is the reference axis) a charged polymer is expected to move through a gel matrix (which provides an entanglement network) according to the rules of reptation. However, standard reptation theory does not allow for a prediction of the drift velocity v beyond the linear response regime of small fields or very long polymers where

$$v \propto DEL. \tag{2.8}$$

Here D is the diffusion coefficient of an entangled polymer, predicted by reptation theory to scale

$$D \propto 1/L^2 \tag{2.9}$$

with length. The extended Rubinstein–Duke model is an asymmetric three-state exclusion process (1.10) with $D_{AB} = D_{BA} = 0$ and open boundaries. Exact and rigorous results [114, 115] confirm the predictions (2.8), (2.9). Moreover, simulations at high fields yield the drift velocity in the nonlinear regime [116] which is in good agreement with experimental data [117]. At sufficiently high fields the model exhibits spontaneous symmetry breaking in the orientation of the polymer chain [118]. The asymptotic behaviour (2.9) of the diffusion coefficient has also been proved to remain valid in the presence of quenched kinematic disorder which is described as a generalized Rubinstein model with many conservation laws where particles have their individual hopping rates [119].

A long-standing mystery in reptation theory has been the asymptotic behaviour of the viscosity η of a polymer melt which is expected to scale asymptotically [108, 109]

$$\eta \propto L^3. \tag{2.10}$$

However, experiments consistently give higher value $\approx$3.4 of the scaling exponent. Doi had suggested this to be a finite-size effect due to tube-length fluctuations [108]. That tube-length fluctuations lead to an increased effective exponent could be confirmed by a careful numerical analysis of the Rubinstein–Duke model [120]. Also details of the end-segment dynamics were shown to have significant non-universal impact on finite-size behaviour of the viscosity and the diffusion coefficient [121].

Spin relaxation. By interpreting occupation numbers as classical spin variables $s_k = 1 - 2n_k$ the ASEP describes biased Kawasaki spin-exchange dynamics [122] for the one-dimensional Ising model at infinite temperature. At finite temperature, biased Kawasaki dynamics correspond to an exclusion process

$$\begin{aligned} XA0Y &\to X0AY \quad \text{with rate} \quad D_r^{XY} \\ X0AY &\to XA0Y \quad \text{with rate} \quad D_l^{XY}. \end{aligned} \tag{2.11}$$

with next-nearest neighbour interaction depending on the occupation X, Y. With some constraints on the rates [67] this model has a stationary distribution

$$P(\underline{n}) = \frac{1}{Z}\,\mathrm{e}^{-\beta(E(\underline{n})+\mu N)} \tag{2.12}$$

which is an Ising measure with energy $E = -J\sum_k n_k n_{k+1}$ and a magnetic field μ which plays the role of a chemical potential in the lattice gas interpretation of the variables. At sufficiently low temperatures the particle (=spin) current becomes a non-convex function of the density (=magnetization) and unusual phenomena such as splitting of shock fronts which separate regions of different density can be observed [7, 123]. In spin language the three-state model (1.10) describes dynamics of a classical spin-1 system.

Interface growth. It was already realized in the 1980s that the ASEP describes the dynamics of a fluctuating interface by considering the spin variables as local discrete slopes of an interface on a two-dimensional substrate [124, 125] (figure 2). Hopping of a particle to the right between sites $k, k+1$ corresponds to the random deposition of a particle on site k of the dual growth lattice, hopping to the left to an evaporation (figure 2). One thus obtains a growth model in the universality class of the one-dimensional KPZ equation [126], reviewed in [44]. We stress that the mapping is not one-to-one. Since in the exclusion presentation only the local slopes enter, the information about the actual height of the interface gets lost. One can keep track of the height by introducing an extra random variable for the local height at some reference point k_0, which is increased (decreased) by 2 units whenever a particle hops across the bond k_0, k_0+1 to the right (left). The steady-state current of the ASEP then gives the average growth velocity, while fluctuations of the current measure fluctuations in the local interface height. The extension of this mapping to the generalized exclusion process (1.10) is obvious, one obtains a system where local height differences may take values $0, \pm 1$. Some growth dynamics considered below have the property that particles cannot be chipped off a complete layer. This corresponds to a hidden conservation law, which cannot be expressed in terms of particle occupation numbers alone.

3. Steady states and hydrodynamic limit

3.1. Steady states for driven diffusive systems

As has become clear above, the stationary behaviour, i.e. the state the system evolves into, is the first question to be addressed in the investigation of driven diffusive systems[3]. In this paper we are concerned with the behaviour of translational-invariant systems, defined either on a finite lattice with periodic boundary conditions or on the infinite integer lattice $\mathbb{Z}$. Consequently, we shall investigate stationary distributions which are either translation invariant or where translation invariance is spontaneously broken.

It is important to bear in mind that entirely different dynamics may have the same stationary distribution. Indeed, for any given distribution one may always construct some equilibrium dynamics (obeying reversibility) using the principle of detailed balance. Moreover, a strongly

[3] This is formally analogous to investigating the equilibrium behaviour of a many-body system and hence in the mathematical literature stationary states are often referred to as equilibrium states, even though the presence of macroscopic currents prevents the applicability of the usual notions of equilibrium statistical mechanics such as reversibility and detailed balance. We note, however, that as long as only a stationary distribution is concerned—without reference to the stochastic dynamics for which the distribution is stationary—it is convenient to use notions borrowed from equilibrium statistical physics such as partition function or canonical/grand-canonical ensembles respectively.

nonequilibrium system may have the stationary distribution of some equilibrium model, an example being the KLS model (2.11). Therefore, equality of stationary ensembles for different systems has no implications whatsoever on the dynamical properties of these models. We also remark that simple dynamical rules may result in stationary distributions with a complicated structure and long-range correlations (see below), while complicated dynamical rules may very well lead to simple stationary distributions. Some stationary distributions for one-species models have been reviewed in the introduction, here we focus on the two-species ASEP.

Fortunately, not only the single-species ASEP but also many two-species stochastic particle systems of interest have simple stationary distributions, the simplest being product measures with stationary probabilities of the form

$$P \propto \mathrm{e}^{\mu^A N^A + \mu^B N^B}. \tag{3.1}$$

Here $N^{A,B} = \sum_k n_k^{A,B}$ are the conserved total particle numbers of each species (for one-component systems one has $n_k^B = 0$) and $\mu^{A,B}$ are the corresponding chemical potentials. For fixed N^A, N^B all configurations are equally likely. In equilibrium, such measures correspond to non-interacting systems.

By writing the master equation for the stochastic dynamics in the quantum Hamiltonian formalism [4] it is straightforward to determine what dynamics have stationary product distributions. In this formalism, a product measure is represented by a tensor product vector $|P\rangle = |p\rangle^{\otimes L}$. Each factor $|p\rangle$ in the tensor product has as components the probabilities of finding a lattice in a given state. Hopping events between sites $k, k+1$ are generated by a local stochastic matrix h_k acting non-trivially only on the terms $k, k+1$ in the tensor product. The full time evolution is generated by the stochastic Hamiltonian $H = \sum_k h_k$ and the stationarity condition for $|P\rangle$ reads

$$H|P\rangle = 0. \tag{3.2}$$

Because of translational invariance a stationary product measure therefore satisfies the relation

$$h_k|P\rangle = (d_{k+1} - d_k)|P\rangle \tag{3.3}$$

with an arbitrary matrix d_k acting non-trivially only on site k. This relation is usually very easy to verify. A similar approach can be chosen for more complicated measures, e.g., Ising measures with stationary probabilities of the form

$$P \propto \exp\left(-\beta \sum_k \left(J^{AA} n_k^A n_{k+1}^A + J^{AB} n_k^A n_{k+1}^B + J^{BB} n_k^B n_{k+1}^B\right) + \mu^A N^A + \mu^B N^B\right). \tag{3.4}$$

Given these measures the current can be calculated exactly as a function of the densities ρ^A, ρ^B via the invertible relationship between the chemical potentials and the densities. For the two-state ASEP (1.10) one has a product measure on the parameter manifold defined by

$$D_{0A} - D_{A0} + D_{B0} - D_{0B} + D_{AB} - D_{BA} = 0. \tag{3.5}$$

For a K-species ASEP with rates D_{XY} for the hopping process $XY \to YX$ there are $(K-1)(K-2)/2$ conditions for the existence of a product measure [127]

$$D_{0X} - D_{X0} + D_{Y0} - D_{0Y} + D_{XY} - D_{YX} = 0. \tag{3.6}$$

In the quantum Hamiltonian formalism the stationary distribution is the ground state vector of the associated quantum spin chain. For integrable models [128, 129] one may use the Bethe ansatz and symmetry properties [18] for the explicit construction of stationary states which are not simple product measures. Using the Bethe ansatz has not been attempted yet for this class of models. A more popular method is the application of the matrix product ansatz, reviewed in [130]. In this approach one defines a product measure with matrix

entries D_m rather than c-numbers as stationary weights for finding a given site in state m. The matrices D_m together with a set of auxiliary matrices [131, 132] have to satisfy algebraic relations which are obtained from requiring the matrix product state to satisfy the stationarity condition (3.2). This leads to algebras with quadratic relations [20, 78, 127, 134–138].

The matrix product construction is equivalent to writing the stationary distribution of the lattice gas in terms of a transfer matrix $C = \sum_{m=0}^{K} D_m$ of some n-state equilibrium system, determined by the representation of the algebra, in particular its dimension n which may be finite or infinite. It is clear that finite-dimensional representations correspond to stationary states with exponentially decaying correlations unless the largest eigenvalue of C is degenerate. The usual product measures (complete absence of correlations) correspond to one-dimensional representations. For a detailed review, see [11, 130]. The approach can also be extended to describe the full time evolution and hence yield time-dependent probabilities of the system evolving from some nonstationary initial distribution [131–133]. Popkov *et al* have identified the parameter manifold for which the dynamics of the two-species ASEP can be solved using the dynamical matrix product ansatz [129]. Such models are all integrable in the sense of being associated with an integrable vertex model. The more general models of [127, 138] for which only the stationary distribution can be constructed with matrix products also include non-integrable models.

3.2. Steady states with one B-particle

A series of intriguing results have been obtained for two-species systems (1.10) with just one particle of type B. Conditioning on having a second-class particle at some given site and calculating the probability of finding a first-class particle at distance r yields the density profile as seen from the shock position, defined by the position of the second-class particle. The density approaches its asymptotic shock densities $\rho_{1,2}$ at an exponential rate, given in a non-trivial way by the hopping asymmetry D_r/D_l and the densities $\rho_{1,2}$ [85, 139]. For a special value of the asymmetry one has Bernoulli measures with densities $\rho_{1,2}$ to the left and right respectively. This is the result of a q-deformed $SU(2)$-symmetry of the Heisenberg quantum Hamiltonian that generates the time evolution of the process. For this value of the asymmetry (or, equivalently, arbitrary asymmetry, but special density ρ_2) the time evolution of the shock measure has been calculated exactly for both the continuous-time ASEP [87] and a discrete-time variant [37]. The shock position performs a lattice random walk with rates given by the currents and densities in the two branches of the shock. Two consecutive shocks with densities $\rho_{1,2,3}$ which can be defined by two second-class particles form a bound state with finite mean distance and exponentially decaying distance distribution if a condition on $\rho_{2,3}$ originating in the q-deformed $SU(2)$-symmetry is met [140]. Generically, two second-class particles form a weak bound state with infinite mean distance and algebraically decaying probability $p(r) \propto r^{-3/2}$ of being a distance r apart [85, 139].

Using the algebra arising from the stationary matrix product ansatz Mallick [94] has studied the two-species ASEP (1.10) with rates

$$D_{A0} = 1 \qquad D_{B0} = \alpha \qquad D_{AB} = \beta. \tag{3.7}$$

All other rates are zero. For $\alpha = 1, \beta = 0$ the 'impurity' particle B corresponds to a tracer particle, for $\alpha = \beta = 1$ it is a second-class particle. This model describes 'cars' (A-particles) and 'trucks' (B-particles), with a passing rate β. For $\alpha < 1$ and $\beta < \alpha$ a single truck acts like an impurity, hindering the motion of cars. The current of A-particles, the velocity of the impurity and the density profile as seen from the impurity have been calculated exactly [94]. The system with a single impurity exhibits an interesting phase diagram as a function of the

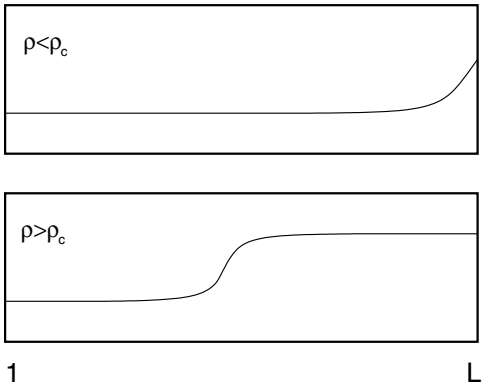

Figure 3. The main features of density profiles in a periodic system in the presence of a localized defect or a mobile impurity at site L. Below the critical density there is an exponentially decaying profile behind the disturbance, above the critical density one has a macroscopic 'traffic jam' extending over a finite fraction of the total length L. Depending on the specific system, the density profile may have some extra structure (not shown here) around the disturbance. The position of the shock at the end of the high-density domain fluctuates.

hopping rates. In one of the phases the system develops a stationary shock for sufficiently large density $\rho > \rho_c$, analogous to a traffic jam building up behind a slow vehicle. One has coexistence of a low-density domain and a high-density domain, separated by a domain wall (figure 3). The diffusion constant of the impurity has also been calculated exactly [90, 141].

Lee *et al* [95] considered the model (1.10) with rates

$$D_{A0} = 1 \qquad D_{0B} = \gamma \qquad D_{AB} = 1/\beta \tag{3.8}$$

corresponding to oppositely moving particles (slow-moving 'trucks' for $\gamma < 1$) which interact upon encounter. In the presence of a single truck (impurity) the average speed of cars (and hence the current), the speed of the truck and the density profile of cars have been calculated using again the same stationary three-species algebra [95]. One obtains a phase diagram with a transition to a jammed phase at a critical density $\rho_c = 1/\beta$ (figure 3), with the remarkable property of having the same statistical properties as a deterministic ASEP with a fixed impurity [142]. In a finite system of site L the position of the (microscopically sharp) domain wall fluctuates over a region of length $\propto \sqrt{L}$. In a system with two trucks, a weak bound state in the traffic jam phase is formed. Note that the model (3.8) is equivalent to (3.7) by exchanging $B \leftrightarrow 0$. However, in this mapping a single truck corresponds to a single vacancy, a scenario not studied in [94].

Arndt *et al* [143] introduced a model of type (3.8) with

$$D_{A0} = \lambda \qquad D_{0B} = \lambda \qquad D_{AB} = q \qquad D_{BA} = 1. \tag{3.9}$$

For finite densities of both particle species this model is reviewed below. Jafarpour [96] considered the presence of a single B-particle and calculated exactly for $\lambda = 1$ the speed of A-particles and the impurity and the density profile of A-particles. As in the model of Lee *et al* there is a phase transition from a free-flowing to a jammed phase, here at a critical density $\rho_c = q/2$ (figure 3).

We remark that the jamming transition seen in these three versions of the two-species ASEP may be regarded as a kind of condensation transition where a finite fraction of A-particles condenses into a macroscopic block trailing the moving impurity. A similar transition also occurs in the usual ASEP with a fixed blockage [142, 144, 145], corresponding to an immobile

B-particle and passing event $AB0 \to 0BA$ rather than $AB \to BA$. Hence single particles in a system with two conservation laws play a role somewhat similar to local inhomogeneities in a system with one conservation law. Note that for a fixed blockage and for the model of Mallick the jammed phase exists between two critical densities ρ_c^-, ρ_c^+. Lee *et al* and Jafarpour respectively report only a lower critical density, below which the system is in the free-flow phase.

3.3. Hydrodynamic limit for finite densities

The link between the stationary state and the dynamics is established by the continuity equation which relates the change in the local order parameters to the currents and other equations for the dynamics of the various nonconserved internal degrees of freedom. In order to obtain information about the dynamics on a coarse-grained scale one assumes local stationarity and a sufficiently smooth behaviour of the local order parameter. For driven diffusive systems one may then investigate the dynamics on the Euler scale, i.e., in the scaling limit where the lattice spacing a and time scale τ are sent to 0 such that a/τ remains constant.

By expressing the correlation functions that enter the currents completely in terms of the density one thus obtains a partial differential equation

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x} j(\rho) \tag{3.10}$$

for systems with one conservation law, and system of equations

$$\frac{\partial}{\partial t}\rho^A = -\frac{\partial}{\partial x} j^A(\rho^A, \rho^B) \tag{3.11}$$

$$\frac{\partial}{\partial t}\rho^B = -\frac{\partial}{\partial x} j^B(\rho^A, \rho^B) \tag{3.12}$$

for models with two conservation laws. Using the theory of partial differential equations this yields coarse-grained information about the time evolution of the lattice model. The average occupation of the local density is thus described in terms of a deterministic evolution of a coarse-grained density profile.

From these introductory remarks it has become clear that knowing the stationary currents *exactly* as functions of the density is crucial for calculating the density profile. A mean-field approximation which neglects correlations between different lattice sites is bound to give wrong quantitative results unless the stationary distribution happens to be characterized by the absence of correlations. For systems with a single conservation law, mean-field approximations may yield qualitatively correct behaviour if the mean-field current reproduces local extrema and inflection points of the true current. However, we wish to stress that even short-ranged correlations resulting from Ising-type stationary distributions (2.12) may lead to qualitatively wrong behaviour of the current–density relation. An example is the KLS model (2.11) where the exact current has a local minimum between two symmetric maxima [7, 146] whereas the mean-field approximation yields the current–density relation (1.4) of the ASEP with a single maximum. Hence neglecting short-ranged correlations may yield not only quantitatively but even qualitatively wrong predictions for the most basic dynamical properties of the system, namely the stationary current and the coarse-grained time evolution of the density profile. To conclude, one needs to know which generic features of a current–density relation determine the qualitative behaviour of the solution of the hydrodynamic equation and it must be verified that a mean-field approximation reproduces those features. Otherwise a mean-field treatment of the continuity equation yields no information about the dynamics of the system.

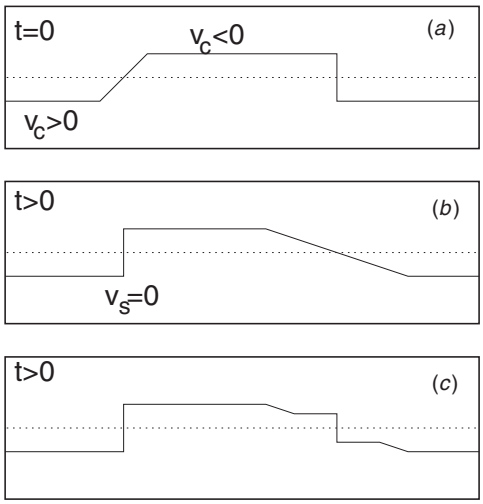

Figure 4. Time evolution of the ASEP on the Euler scale, starting from time $t = 0$ (*a*) with an unstable shock and a region of positive slope. Any region of positive slope evolves into a shock after some time (*b*) because of the flow of localized perturbations (microscopic picture) or characteristics respectively (macroscopic description). The evolution of the unstable shock is not uniquely defined by the viscosity-free hydrodynamics. Both evolutions (*b*) and (*c*) are solutions of the inviscid Burgers equation for initial state (*a*). The physical solution selected by the spatially homogeneous ASEP is shown in (*b*). The time evolution shown in (*c*) corresponds to the ASEP with a defect which is governed by the *same* homogeneous Burgers equation [149].

In some cases approximating an unknown measure by an Ising measure (2.12), (3.4) with short-ranged correlations rather than by a simple product measure (simple mean field) without correlations may bring improvement. In the literature this improved kind of approximation scheme is sometimes called cluster approximation. In the following, we assume that at least the stationary current (if not the full measure) is known exactly.

Nonlinear equations of the form (3.10), (3.11) are known to possess singularities which do not allow for a unique solution of the initial value problem. Almost all initial configurations will develop shock discontinuities where the density jumps from one value to another even if the initial state was smooth. This raises the question of the microscopic properties of the macroscopic solution. Moreover, ambiguities exist even if the original particle problem has a unique stationary state into which the system evolves for all initial states. Thus one is faced with the second problem of selecting the physical solution of the hydrodynamic equation (figure 4).

For a class of systems with one conservation law the transition from the stochastic lattice dynamics to the hydrodynamic equation (3.10) is mathematically well understood [147] and includes also the treatment of shock discontinuities, for a broader overview see [1]. Here we give a more physics oriented account inspired by the desire to derive macroscopic phenomena such as shocks from microscopic behaviour, namely the flow of localized (microscopically) perturbations inside a stationary region. It has turned out that such an approach, originally developed for systems with one conserved density and no internal degrees of freedom, also works for two-species problems, if suitably generalized.

To keep the discussion simple we restrict the review to a system with a convex current–density relation such as found in the ASEP (1.4). In the hydrodynamic limit (3.10) one obtains

for the ASEP the well-known inviscid Burgers equation [148]

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(D_r - D_l)\rho(1-\rho) = -(D_r - D_l)(1-2\rho)\frac{\partial}{\partial x}\rho. \tag{3.13}$$

It is well known that an upward shock in the ASEP constitutes a stable shock whereas the downward shock dissolves into a rarefaction wave. This result can be obtained by using the method of characteristics which traces the motion in spacetime of points of constant density. One introduces a scaling variable $u = x/t$ to obtain from (3.13)

$$u = (D_r - D_l)(1-2\rho). \tag{3.14}$$

This is a weak solution for an initial profile with a down-shock at $x = 0$ (figure 4).

This solution may be obtained in a different way by starting from the lattice continuity equation (1.2) with current

$$j_k = (D_r - D_l)\rho_k(1-\rho_{k+1}) \tag{3.15}$$

obtained from (1.3) by neglecting correlations at all times. The picture underlying this approximation is the assumption of local stationarity where all correlations are sufficiently small. With a Taylor expansion in the lattice constant $a \to 0$ one arrives at the Burgers equation

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x}(D_r - D_l)\rho(1-\rho) + \nu\frac{\partial^2}{\partial x^2}\rho \tag{3.16}$$

with an infinitesimal viscosity $\nu \propto a$. The Burgers equation is integrable using the Hopf–Cole transformation $\rho = \kappa\partial_x \ln w$. This leads to a standard linear diffusion equation for w which has a unique solution for any initial profile. Taking the limit $a \to 0$ in the solution one recovers the physical solutions described above (shock and rarefaction waves) which are realized by the ASEP.

In a different, but for one-species models equivalent, approach for selecting the physical solution one defines an entropy function associated with the conservation law (3.10). The entropy solution yields the physical solution corresponding to the ASEP, for a review, see [1]. With this approach one may also consider the ASEP with a localized defect [149] which has a different solution for a downward shock and which also produces a shock inside a domain with constant density [150].

In order to obtain a physical microscopic picture of how these solutions emerge on a macroscopic scale, we study the dynamics of localized perturbations in a homogeneous stationary environment [4]. The time evolution of such a perturbation on the lattice scale can be probed by examining the dynamical structure function

$$S(k,t) = \langle n_k(t)n_0(0)\rangle - \rho^2 \tag{3.17}$$

which measures the density relaxation of a local perturbation in the stationary state of uniform density ρ. Generally, the width of such a perturbation at $t = 0$ is of the order of the bulk correlation length. The centre-of-mass velocity of the perturbation is given by the collective velocity

$$v_c = \frac{\partial}{\partial\rho}j(\rho). \tag{3.18}$$

One may derive this relation from the shock velocity

$$v_s = \frac{j_1 - j_2}{\rho_1 - \rho_2} \tag{3.19}$$

by taking the limit $\rho_1 \to \rho_2$ of the asymptotic densities of the shock. Note that v_c changes sign at local extrema of the current–density relation. We stress that expression (3.19) follows

from mass conservation and hence no specific assumptions on the nature of the microscopic dynamics are involved. Hence also (3.18) is expected to be generally valid. The only assumption is sufficiently rapidly decaying bulk correlations in the steady state as otherwise the microscopic definition of a shock position becomes questionable, since the shock position cannot be defined on a scale below the bulk correlation length. A direct derivation of (3.18) from the dynamical structure function which uses only translational invariance, the conservation law and decay of correlations is outlined in [4].

The shock velocity for the ASEP follows from (1.4), (3.19) and one finds

$$v_s = (D_r - D_l)(1 - \rho_1 - \rho_2). \tag{3.20}$$

This yields the collective velocity

$$v_c = (D_r - D_l)(1 - 2\rho). \tag{3.21}$$

The origin of the physical solution of the macroscopic time evolution can now be explained from a microscopic viewpoint by imagining that by a small fluctuation a certain amount of mass detaches from the shock and forms a perturbation at a small distance from the shock position. On average this fluctuation will then travel with speed (3.21) where ρ is to be taken as either ρ_1 or ρ_2, depending on whether the fluctuation had originally moved to the left or right of the shock. Equation (3.21) shows in the case of an upward shock that for all shock densities $\rho_{1,2}$

$$v_c^{(1)} > v_s > v_c^{(2)}. \tag{3.22}$$

Hence in the moving reference frame of the shock the excess mass drifts back to the position of the shock and hence stabilizes it.

On the other hand, in the case of an initial downward shock the fluctuating excess mass moves on average away from the shock. Therefore this shock is not stable against fluctuations, in the course of time the shock smears out and develops into a rarefaction wave. In order to predict the macroscopic shape of the rarefaction wave, we assume an initial configuration with a shock with densities ρ_1 and ρ_2 which is composed of many infinitesimal subsequent shocks at various levels of intermediate densities (figure 3). Neither of these shocks is stable, but each slowly dissolving shock at density ρ moves with a speed v_c. Hence we conclude that on the Euler scale (where the spread of a perturbation (2.6) and hence the increasing width of the unstable shock is scaled to zero) points of constant density ρ generally move with speed v_c. From this observation the explicit form of the rarefaction wave can be constructed.

Clearly, this is not a rigorous argument. Support for this picture comes from the hydrodynamic limit. The collective velocity is then nothing but the speed of the characteristics of the corresponding hydrodynamical equation $\partial_t \rho = -\partial_x j$ resulting from the continuum limit of the lattice continuity equation (1.2). In this limit, the criterion (3.22) becomes the defining property of a shock discontinuity [151]. It asserts that the characteristics are moving into the shock. Otherwise, the characteristics yield the rarefaction wave, as rationalized above. For current–density relations which are not globally convex one decomposes a single shock into subsequent small shocks and then applies (3.22) to these minishocks in order to decide on stability. By taking the limit of infinitesimal shocks one recovers in this way the scaling solution of the hydrodynamical equation obtained from the method of characteristics [7, 152].

For systems (3.11) with two or more conservation laws there is no well-established mathematical theory for the selection of the physical solution in the corresponding lattice gas. In recent work [32] Tóth and Valkó have obtained rigorous results by making use of Yau's relative entropy method [153] which essentially proves that a product measure

with time-dependent local densities evolving according to the solution of the hydrodynamic equation converges to the true measure in the sense of relative entropy of the two measures. This approach works for systems with a stationary product measure until a shock has formed, provided some particular identities hold which relate the macroscopic fluxes in the hydrodynamic PDE. These identities are reminiscent of the Onsager reciprocity relations. The systems studied in [32] are models with generically more states than conservation laws. They include a family of two-species ASEPs with parameters satisfying relation (3.5) guaranteeing the existence of a stationary product measure and which have a natural interpretation as growth models. A two-species zero-range process has been studied by Grosskinsky and the corresponding hydrodynamic equations have been established [154].

The rigorous approach of [32] is rather powerful but in its current state fails as soon as shocks develop. On the other hand, shock waves and special rarefaction waves have been analysed nonrigorously from a microscopic viewpoint by studying the flow of perturbations and correspondingly extending the physical arguments presented above to two-species systems [155]. One has to study two perturbations in each conserved density which due to the interaction are forced to move with the same velocity. Generalizing the analysis of the dynamic structure function to two conservation laws one finds that the main difference in the case of one conservation law is the evolution of two distinct pairs of perturbations out of a single pair. Each pair moves with collective velocities $v_c^{\pm}$ given by the eigenvalues of the Jacobian

$$\mathcal{D} = \begin{pmatrix} \dfrac{\partial}{\partial \rho^A} j^A & \dfrac{\partial}{\partial \rho^B} j^A \\ \dfrac{\partial}{\partial \rho^A} j^B & \dfrac{\partial}{\partial \rho^B} j^B \end{pmatrix}. \tag{3.23}$$

Corresponding to the two pairs of perturbations a single shock splits into two separate shocks, leaving the system in stationary regimes of three distinct densities, namely the left and right shock densities $\rho_{1,2}^{A,B}$ enforced by the initial state and self-organized intermediate shock densities $\tilde{\rho}^{A,B}$. The equality of each pair of shock velocities (given by the general expression (3.19) applied to each single shock) and the requirement that the velocity of the left shock v_s^L must be smaller than the velocity v_s^R of the right shock determine the intermediate densities $\tilde{\rho}^{A,B}$. Requiring that all perturbations be absorbed in the shock one arrives at the condition

$$v_c^{\pm}(\rho_1) > v_s^L > v_c^-(\tilde{\rho}) \qquad v_c^+(\tilde{\rho}) > v_s^R > v_c^{\pm}(\rho_2) \tag{3.24}$$

for shock stability in driven diffusive systems with two conservation laws. Violating one of these conditions leads to rarefaction waves which have partially been described [155]. The general features discussed here are confirmed by Monte Carlo simulation of a two-lane model with a conserved density on each lane related to the models of [30, 31], but with periodic boundary conditions and different choice of hopping rates respectively. A complete description of the evolution for all possible initial states has not yet been achieved.

4. Critical phenomena

It is well known that in thermal equilibrium one-dimensional systems with finite local state space and short-range interactions do not exhibit phase transitions at positive temperatures, only at $T = 0$ may long-range order exist. From a dynamical viewpoint there are no thermal fluctuations at $T = 0$ in a classical system. In terms of a stochastic process it means that all transition rates are zero. Conversely, if a transition rate is non-zero, some dynamics—not

necessarily satisfying detailed balance—is going on and it has been conjectured that quite generally a system with strictly positive transition rates and local interactions can have at most one stationary distribution, which is often rephrased by saying that there can be no phase transition in a one-dimensional system with strictly positive rates. One has in mind an infinite system since in a finite system dynamics with strictly positive rates are always ergodic and the conjecture is trivially true.

To rationalize the conjecture one imagines, in the simplest case, two potentially stationary distributions characterized by different values of the order parameter. An example is the Ising model where the order parameter is the magnetization, which can take two different values below the critical temperature in two or higher dimensions. The reasoning behind the positive rate conjecture is the difficulty of imagining a local mechanism that eliminates islands of the minority phase (created constantly by thermal fluctuations in a region where the other phase dominates) since in one dimension energetic effects due to line tension play no role. A local mechanism cannot detect the size of a minority island, therefore such an island can grow indefinitely and destroy the majority phase. Since noise (implied by strictly positive rates) can always create such islands there seems to be no possibility of keeping the majority phase stable against fluctuations. In a certain 'natural' class of systems with nearest neighbour interaction this conjecture has been proved rigorously some time ago [156].

Therefore it came as a surprise that Gacs constructed a model on the infinite lattice which violates the positive rate conjecture [157, 158]. However, both the model and the proof that there is a phase transition are rather complicated [159], requiring either a very large local state space or a very large interaction range, and the quest for simple models with this property continues to stimulate research. As a guideline, we note that the conjecture is clearly true for dynamics satisfying detailed balance with respect to a local interaction energy as in this case the stationary distribution is just the usual equilibrium distribution and the argument underlying the positive rate conjecture applies. Hence one should look for models that either violate detailed balance or have a nonlocal interaction energy, but local dynamics.

We address the question of phase transitions in a broader sense by asking whether phenomena associated with phase transitions such as divergent length scales or spontaneous symmetry breaking may occur far from equilibrium. Divergence of some correlation length does not necessarily require the existence of more than one stationary distribution for a given set of system parameters. On the other hand, one could have a parameter range with two or more stationary distributions which are not related by any symmetry, but the transition into this regime would be associated with a divergent length scale.

We restrict ourselves to driven diffusive systems with one or two conservation laws, thus skirting the issue of phase transitions in nonconservative models, addressed in [103, 160, 161], and also avoiding systems kept out of equilibrium without having a current in the conserved density. This could be achieved, e.g., by coupling *symmetric* hopping dynamics to heat baths of different temperatures, a scenario not envisaged in the context of driven diffusive systems. Also nonconservative processes with absorbing states such as the contact process [3, 14, 15] fall in the class of systems not considered here. We refer to models of this type only where it serves to illuminate the properties of closely related driven systems.

By definition, conservative systems have a continuum of stationary states (characterized by the value of the order parameter) and hence the critical phenomena we review concern transitions between different stationary distributions at the same value of the order parameter and coexistence of macroscopic stationary domains where the order parameter takes different values. The domain walls separating these domains are the shocks discussed in the previous section. Hence the stability of domain walls is intimately connected with the existence of phase separation.

4.1. One conservation law

In a system with a conserved density the positive rate conjecture does not apply by definition, as transitions violating density conservation have zero rate. Hence there seems to be no reason to pursue the question of existence of phase transitions in conservative systems. Moreover, the discussion of shock stability presented above shows that a stable domain wall separating regions of different values of the order parameter may exist, thus actually suggesting the possibility of phase coexistence and long-range order. However, by the same reasoning it is clear that in a system with a generic current–density relation one cannot have two domain walls which would be necessary to have macroscopic phase separation in a translation-invariant system: if, say, an upward shock from density ρ_1 to density ρ_2 at the left boundary of the region of higher density ρ_2 is stable by the criterion (3.22), then the downward shock ρ_2 to ρ_1 at the other boundary of the region of higher density would be unstable by virtue of the same criterion. Thus one is forced to conclude that phase coexistence in a system with one conservation law may exist only in the absence of translation invariance. This is indeed known in systems with open boundaries where external particle reservoirs enforce regions of constant boundary densities $\rho_{1,2}$, separated by a single stable domain wall [19, 20, 59]. Similarly in a periodic, but not translation-invariant system with a defect one may have phase separation since in this set-up the defect may stabilize the intrinsically unstable shock. Hence, stability of a shock in a system with one conservation law does *not* constitute a violation of the zero-rate conjecture for a translation-invariant system. Indeed, it has been suggested that one-dimensional driven diffusive systems do not exhibit long-range order in their steady states [83].

Yet, several translation-invariant models with one conservation law and short-range interactions which exhibit a robust phase transition were discovered. A simple, but non-generic example is a growth model (which can be mapped to a driven diffusive system according to the strategy explained in section 2) where a roughening transition from a phase with a smooth interface to a phase with a rough interface occurs [162, 163]. This class of models is non-generic in so far as there is an intrinsic maximal growth velocity of the interface, enforced by a discrete-time parallel update. The smooth phase and hence the phase transition disappears if the limit of continuous time is taken in these models.

Addressing the possibility of a roughening transition in systems with continuous time evolution Alon *et al* [164, 165] proposed a two-species ASEP with rates

$$D_{A0} = D_{0B} = (1-q)/2 \qquad D_{0A} = D_{B0} = D_{BA} = q \tag{4.1}$$

and annihilation/creation rates

$$D_{00} = D_{BA} = q \qquad D_{AB} = 1-q \tag{4.2}$$

for the transitions $00 \leftrightarrow AB$ and $BA \rightarrow 00$ respectively. These dynamics lead to a single conserved 'density' $S = N^A - N^B$. In the mapping to a growth model A (B) represent a local slope 1 (-1) and 0 represents slope 0. In order to ensure periodic boundary conditions also in the interface representation the model has been studied for $S = 0$. For small $q < q_c$ there is a smooth phase where a local mechanism eliminates islands in a flat region, since islands are formed with boundaries that are biased to move towards each other. This mechanism applies for islands of all sizes (except completed layers) and hence leads to a smooth interface, consisting mainly of vacancies in the lattice gas picture. Above $q_c \approx 0.189$ the creation of new islands overcompensates the disappearance due to their intrinsic tendency to shrink and the system is in a growing rough phase with a finite fraction of particles. The growth model is in the universality class of the KPZ equation which is represented by the standard ASEP. The critical behaviour at q_c which is related to directed percolation is discussed in detail in

[164, 165] where also a version of the model without constraint on the local slope (corresponding to the absence of exclusion in lattice gas language) is discussed. There is spontaneous symmetry breaking in the smooth phase which can be quantified by introducing either a colouring scheme, giving vacancy clusters between A, B pairs a colour, or by introducing a nonconserved order parameter M that makes use of the interface representation. The critical exponent θ associated with the vanishing of the order parameter by approaching the critical point from below,

$$\langle |M| \rangle \sim (q_c - q)^{\theta} \tag{4.3}$$

is a new exponent, not yet understood in the framework of directed percolation.

For this transition to exist it is crucial that no chipping of particles from the interface in a locally flat environment may occur. This is ensured by imposing a kinetic constraint on the local dynamics, namely setting the rate $\tilde{D}_{00}$ for the process $00 \to BA$ to zero. However, this process does not violate the single conservation law. Hence this model does not provide a counterexample against the zero-rate conjecture, as applied to systems with one conservation law. By regarding the various local transitions of this strongly nonequilibrium process as induced by thermal activation from heat baths at different temperatures, one is led to conclude that the local mechanism that guarantees bounded growth of regions of the minority phase is achieved at the cost of requiring a zero-temperature condition on the chipping process.

In the presence of chipping with a rate $\tilde{D}_{00} = p$ the interface attains a negative stationary growth velocity for some value of q that depends on p. An interesting phenomenon then occurs if the dynamics of the interface is constrained by the minimal height condition that $h_i \geqslant 0$ for all times and all lattice points i [166]. For negative growth velocity, the interface is driven towards the hard wall located at the height level $h_i = 0$ which the interface cannot penetrate. In the special case of $p = 1 - q > \frac{1}{2}$ the model satisfies detailed balance with respect to the energy

$$E = \sum_{i=1}^{L} h_i \tag{4.4}$$

which is the area under the interface. The stationary probability of finding an interface configuration $\underline{h} = (h_1, \ldots, h_L)$ is given by

$$P(\underline{h}) = (q/(1-q))^{E(\underline{h})}/Z_L \tag{4.5}$$

with the partition function $Z_L = \Sigma_{\text{config}}(q/(1-q))^E$. At $q = \frac{1}{2}$ the interface has mean velocity zero, for $q > \frac{1}{2}$ the interface grows. Hence expression (4.5) diverges in time and becomes meaningless as a stationary distribution for $q \geqslant \frac{1}{2}$. In the growth regime the interface roughens, with dynamical behaviour in the ubiquitous KPZ universality class. For $q < \frac{1}{2}$ the interface is bound to the hard wall and hence is smooth. Close to the transition point exact analysis of the partition function yields an occupation density σ of the bottom layer $h = 0$ and a width w diverging as [166]

$$\sigma \sim (q_c - q)^1 \qquad w \sim (q_c - q)^{1/3}. \tag{4.6}$$

The unbinding of the interface at $q_c = \frac{1}{2}$ is analogous to a wetting transition. We remark that this model has a zero-rate constraint by not allowing the interface to penetrate the bottom layer.

The physics described by these two models can be captured in a generalized KLS model (2.11). In this model spontaneous symmetry breaking due to the absence of chipping and conservation of minimal height with the resulting wetting transition can be studied without reference to the height variable.

Koduvely and Dhar considered the symmetric KLS model with rates $D^{XY} := D_r^{XY} = D_l^{XY}$ for the hopping event $XA0Y \leftrightarrow X0AY$ [167]. The analogue of the chipping rate in the two-species model is the hopping rate D^{A0}. Setting $D^{A0} = 0$ automatically leads to conservation of minimal height or, more precisely, conservation of the height level of a completed layer. It is not necessary to stop the dynamics by a separate rule involving the local height. In contrast to the previous model of Hinrichsen *et al*, however, the interface always remains anchored to the minimal height level at some random position. Hence the properties of the wetting (unbinding) transition are described by this model only below the critical point in the dry (bound) state.

Careful numerical analysis of the symmetric model [167] indicates subdiffusive critical dynamics of the unbiased interface with a dynamical exponent $z \approx 2.5$, as opposed to the usual dynamical exponent $z = 2$ of the unbiased Edwards–Wilkinson interface modelled by the symmetric KLS model for any $D^{A0} > 0$. The physics of the biased interface was studied by Helbing *et al* [168] by considering asymmetric rates

$$D_r^{AY} = r \qquad D_r^{0Y} = q \qquad D_l^{0Y} = p \qquad D_l^{AY} = 0. \tag{4.7}$$

with $p = 1 - q - r$. For $r = 0$, the system satisfies detailed balance with respect to the measure (4.5). Below the wetting transition $q < \frac{1}{2}$ one has the exact exponents (4.5). At the critical point the interface is rough and one expects a dynamical critical exponent as measured in the related model of Koduvely and Dhar. Above the critical point the interface would grow, but cannot detach from the minimal height level due to anchoring. Hence the measure (4.5) is stationary for all q. The stationary interface has a cusp, the anchoring point is random. This is an example of spontaneous breaking of translational invariance. In the particle picture the steady state is a shock measure with extremal limiting densities $\rho_1 = \epsilon, \rho_2 = 1 - \epsilon$ with a sharp downward shock at some random lattice site k. In a finite system, the small quantity ϵ and hence the particle current are exponentially small in system size L. The random anchoring point k moves with a speed also exponentially small in system size. This structure describes a strongly phase-separated state with an essentially empty region and an essentially full region. For this to be valid it is not necessary to require a total average density $\rho = \frac{1}{2}$. It is easy to understand this steady state directly from the microscopic dynamics. The right edge of the occupied domain, i.e. the right-hand shock, is stable because a hopping of the rightmost particle is exceedingly unlikely since the transition $AA0 \to A0A$ is forbidden and the configuration $0A0$ where hopping is allowed is exceedingly unlikely for the rightmost particle (exponentially small in system size). On the other hand, at the left boundary of the domain the system essentially behaves like an ordinary ASEP which has a stable upward shock in the direction of motion.

For $r > 0$ the interface is not anchored anymore, but the minimal height condition for any completed layer is still conserved. Along the line $r = q$ one observes a transition from a rough growing interface (finite particle current $j > 0$ for $q > q_c \approx 0.1515$) with KPZ dynamics to a smooth interface with spontaneously broken symmetry. The symmetry breaking can be quantified in terms of the nonconserved order parameter

$$\tilde{M} = \sum_k (-1)^k n_k. \tag{4.8}$$

This quantity measures the difference of sublattice densities between the even and odd sublattice respectively. In the language of spin systems this is the staggered magnetization, playing the role of the order parameter for antiferromagnetic systems. Below the critical point, particles accumulate either on the even or on the odd sublattice respectively. Both happen with equal probability, but a transition between both kinds of configurations occurs on a time scale that diverges exponentially in system size. This is the signature of spontaneous symmetry

breaking in a finite system. An explicit calculation of the exponentially large transition time is possible in the vicinity of the line $q = 0$ [169]. No reference to the height variable is necessary for measuring $\tilde{M}$. Numerical investigation of the model shows that it is in the same universality class as the growth model of Alon *et al* [168].

In neither of the models discussed above is the positive rate condition satisfied. Only kinetic constraints imposed by vanishing rates, analogous to the zero-temperature condition on phase transitions in equilibrium, may lead to a vanishing current in which case the stability argument for domain walls does not apply and phase transitions can occur. Below we shall present an independent argument that suggests conditions under which kinetic constraints lead to phase separation. Hence so far there is no known simple model with a single conservation law that violates the positive rate conjecture.

4.2. Phase separation in two-species ASEPs

The exact and numerical analysis of steady states of one-species systems reviewed above has revealed that phase separation in systems defined on a finite ring or on $\mathbb{Z}$ may occur if one or more of the following conditions are satisfied:

(I) There are spatially localized defects reducing the mobility of particles.
(II) Single particles of a different species act as mobile blockages.
(III) The dynamics have kinetic constraints arising from a nonequilibrium zero-temperature condition.

The last condition leads to *strong phase separation* in the sense that one domain is fully occupied whereas the other domain is entirely empty. The current in the phase-separated state vanishes exponentially in the size of the particle domain, the separated state exists at any total particle density. Conditions (I) and (II) may lead to strong phase separation, but allow also for a soft phase separation between domains of different densities. This phenomenon sets in only for densities above some critical density ρ_c. The steady-state current is nonvanishing and independent of ρ in the phase-separated state: increasing the density leads to an increase in the size of the high-density domain, but not to a change of the current. In analogy to Bose–Einstein condensation we call the high-density domain a condensate, the transition at ρ_c is referred to as condensation transition. Note that this characterization refers to the thermodynamic limit $L \to \infty$. In a finite system there is either a current exponentially small in system size (case A, strong phase separation) or one has finite-size corrections to the finite bulk current (case B, soft phase separation). We remind the reader that soft phase separation may disappear above a critical density $\tilde{\rho}_c$ and also for a finite density of blocking particles [95]. Strong phase separation is accompanied by spontaneous breaking of translational invariance, except if caused by condition (I) where translational invariance is explicitly broken (figure 5).

Strong phase separation has been found also in homogeneous systems on a ring where none of conditions (I)–(III) is satisfied, but where there is a second species of particles with finite density [30, 143, 170]. Hence we add a further sufficient condition for the possibility of phase separation:

(IV) The system has two or more conservation laws.

We remark that all conditions (I)–(IV) in some way or the other impose local constraints on the dynamics of the driven diffusive system. This appears to be a general requirement for phase separation in generic driven diffusive systems. The size of the local state space and the range of interaction appear to be irrelevant if one of conditions (I)–(IV) is satisfied.

Using a four-state model which is equivalent to a two-lane model with two conserved densities Lahiri and Ramaswamy [30, 171] address the question of phase separation in terms

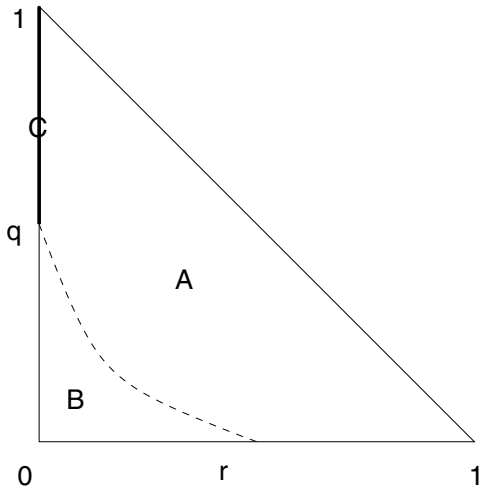

Figure 5. Schematic phase diagram with second-order phase transition line (broken curve) between the disordered phase (A) (growing KPZ interface) and ordered phase (smooth interface) (B) with spontaneously broken Z_2-symmetry and nonvanishing order parameter $\tilde{M}$ (4.8). At $r = 0$, there is a transition at $q = \frac{1}{2}$ with strong phase separation and spontaneous breaking of translational invariance (bold line C).

of the stability of crystals moving steadily through a dissipative medium, e.g., a sedimenting colloidal crystal. In a certain limit (large particle radius or small elastic modulus of the suspension) experiments suggest instability of such a crystal. Numerical analysis of the lattice model, however, reveals a transition to a stable regime, corresponding to strong phase separation. In the simpler two-species ASEP (3.9) or in a more symmetric model with rates [170, 172]

$$D_{A0} = D_{0B} = D_{BA} = 1 \qquad D_{0A} = D_{B0} = D_{AB} = q \tag{4.9}$$

the mechanism for strong phase separation for $q < 1$ is very transparent. Here strong phase separation refers to separation of three pure macroscopic domains, each consisting of essentially only one particle species or empty sites. For simplicity we assume $N^A = N^B$, but this is not necessary for the phenomenon to occur. Prepare a phase-separated block which we symbolically represent by $\ldots 000AAAAAABBBBBB000\ldots$. One observes the following. (i) The $0|A$ interface is stable by the criterion (3.22) since due to the absence of B-particles one has the dynamics of the usual ASEP (with a bias to the right) in the vicinity of this domain wall. (ii) The $B|0$ interface is stable for exactly the same reason (B-particles have a bias to the left). (iii) The $A|B$ interface is stable since in the absence of vacancies B-particles act like vacancies w.r.t. the local dynamics of the A-particles and vice versa. (iv) Since each domain wall is stable (only small fluctuations extended over a finite range of lattice sites evolve at the phase boundaries) the assumption used in the argument remains valid for all times.

It is clear that this model can be extended to an arbitrary number of conserved species and does not require equal density for each particle species. However, for $N^A = N^B = L/3$ the dynamics can be shown to satisfy detailed balance w.r.t. an equilibrium measure with a nonlocal interaction energy of Ising type [172]

$$E = \sum_{k=1}^{L-1} \sum_{l=k}^{L} \left[\left(1 - n_k^A - n_k^B\right) \left(n_l^B - n_l^A\right) + n_k^A n_l^B \right]. \tag{4.10}$$

The corresponding partition is proportional to system size L, rather than e^{fL}, since fluctuations occur only in a finitely extended region around the three domain walls. The 'temperature' associated with equilibrium measure is given by $kT = -1/(\ln q)$. It diverges at $q = 1$ which corresponds to the disordered state of symmetric diffusion of first- and second-class particles. The analysis of the model for $q > 1$ is similar, with the role of A- and B-particles interchanged. The behaviour of the model close to the transition point $q = 1$ has been investigated [173] and some critical exponents have been determined numerically.

The related but distinct model (3.9) of Arndt *et al* has strong phase separation for $q < 1$, with essentially nonfluctuating $0|A$ and $B|0$ domain walls and an $A|B$-interface similar to that in model (4.9). For $q > 1$ the behaviour of the model is more intricate. Numerical and mean-field analysis [174] suggests the existence of soft phase separation up to a critical value $q_c = 1 + 4\lambda\rho/(1 + 2\rho)$. There is a condensate of density 1, but consisting of both species of particles. The other 'fluid' phase has density <1, with particles of both species and vacancies distributed apparently similarly to the disordered phase for $q > q_c$. Note that inside the condensate particles also flow, but with the dynamics of the usual ASEP, as the B-particles act like vacancies in the usual ASEP. The number of A- and B-particles is on average equal and the condensate essentially behaves like the usual ASEP with open boundaries in the maximal current phase, except that the system size M of this ASEP corresponds to the slightly fluctuating cluster size. The stationary current for both particle species is non-zero and approximately given by the value $J = (q - 1)/4$ expected from the ASEP in the maximal current phase. The position of the condensate fluctuates on the lattice. For an unequal average density of A- and B-particles, the scenario as described here remains essentially unchanged, except that the condensate has a finite drift velocity [175].

The exact stationary distribution of the model (3.9) can be calculated using the matrix product ansatz. For the grand-canonical ensemble with equal densities of both particle species it has been shown [176] that the apparent condensation transition is a crossover effect. For sufficiently large lattice one would observe a distribution of clusters, but not a single macroscopic condensate. Using the parameter $a = (1 - q^{-1} - \lambda)/\lambda$ and the fugacity ξ controlling the density the exact current J and the density ρ are given by

$$J(\xi) = \frac{2a^2\xi}{1 + a^2 + 2a(1 + a^2)\xi - (1 - a^2)\sqrt{1 + 4a\xi}} \tag{4.11}$$

$$\rho(\xi) = \frac{a(1 + a)\xi[(1 + a)\sqrt{1 + 4a\xi} - (1 - a)]}{\sqrt{1 + 4a\xi}[1 + a^2 + 2a(1 + a^2)\xi - (1 - a^2)\sqrt{1 + 4a\xi}]}. \tag{4.12}$$

This expression has a very remarkable property: inside the apparently condensed phase at $q = -a = \frac{10}{9}$ the derivative $J'(\rho)$ which enters the Jacobian (3.23) and hence determines the collective velocities of the two-species system has a change of order 1 arising from a change of order 10^{-24}. One would need a lattice of the order of 10^{70} sites to actually observe the breakdown of the condensation and see the full distribution of clusters of various lengths.

Clearly, a crossover scale of this magnitude is of no relevance for the occurrence of soft phase separation in a real system. Any finite sample would exhibit a phase-separated state. However, the huge number 10^{70} characterizing the typical scale of the cluster size distribution is specific for the model (3.9). As shown in [177] the crossover scale depends sensitively on non-universal parameters which are tunable in some models [178]. Hence other models may have parameter ranges with mean cluster sizes of the order, say, 10^5. Such a crossover scale would render numerical results from computer simulations of the steady state for a real quasi-one-dimensional system ambiguous: a computer simulation of a realistic model for a real system with less than 10^5 particles could predict soft phase separation, whereas actual

experiments done on a macroscopic sample with more than 10^5 particles could yield the contradictory result that there is only a disordered phase. Yet, one could not conclude from this observation that the model is inappropriate to describe the real system since in a smaller experimental sample of the same system computer simulations and experimental observations may agree.

We remark that a technical assumption in the exact calculation of Rajewsky *et al* [176] has been proved in [179]. The validity of the result also in the canonical ensemble has been challenged [175], even though exact analysis of the fluctuations in the particle density strongly suggests that there is no true condensation in the canonical ensemble [176]. In another model introduced by Korniss *et al* a two-lane extension of a three-species driven system was studied [180, 181]. It has been suggested that while for this model the one-lane system does not exhibit phase separation [182], this phenomenon does exist in the two-lane model. The studies rely on numerical simulations of systems of length up to 10^4. However, no theoretical insight is available as to why phase separation in this two-lane model should persist in the thermodynamic limit.

To conclude these general considerations we note that in many real quasi-one-dimensional systems the particle number is in the range 10–10^4. Simplified models of such systems are accessible to numerical simulations of the steady state. This raises the further question to what extent phase separation in finite systems is obscured by a too large intrinsic width of the domain wall separating the condensed domain from the 'fluid' low-density domain. While quantitative predictions for the relaxation modes resulting from the coarse-grained domain wall theory in single-species systems with open boundaries have been verified numerically for the ASEP on small lattices of only $\mathcal{O}(10)$ sites [183] there is no systematic finite-size study of soft phase separation.

4.3. Criterion for phase separation

In view of the exact analysis of the model (3.9) it is clear that numerical evidence for soft phase separation may be rather subtle and indeed be misleading. It would thus be of great importance to find other criteria, which could distinguish between models supporting phase separation from those which do not. Phase separation is usually accompanied by a coarsening process in which small domains of, say, the high-density phase coalesce, eventually leading to macroscopic phase separation. This process takes place as domains exchange particles through their currents. When smaller domains exchange particles with the environment with faster rates than larger domains, a coarsening process is expected, which may lead to phase separation.

An approach that quantifies this mechanism and yields a criterion for phase separation in terms of the current leaving the domains is proposed by Kafri *et al* [184]. The current criterion is readily applicable even in cases which cannot be decided by direct numerical simulations. In order to explicitly state the criterion one distinguishes systems with a vanishing current inside a finite domain of size n

$$J_n \to 0 \quad \text{(case A)} \tag{4.13}$$

from systems with finite-size corrections to a finite asymptotic domain current J_∞ of the form

$$J_n = J_\infty(1 + b/n^\sigma) \quad \text{(case B)}. \tag{4.14}$$

to leading order in $1/n$. For simplicity, we assume here domains with vanishing drift velocity in which case the current inside the domains equals the outgoing current. More generally, one has to distinguish the two currents leaving the cluster at the right and left boundary respectively.

For $b > 0$ the current of long domains is smaller than that of short ones, which leads to a tendency of the longer domains to grow at the expense of smaller ones. The current criterion asserts that phase separation exists only in the following cases [184],

$$J_n \to 0 \qquad \text{for} \quad n \to \infty \quad \text{(case A)} \tag{4.15}$$

$$J_n \to J_\infty > 0 \quad \text{(case B)} \tag{4.16}$$

for either $\sigma < 1$ and $b > 0$ or $\sigma = 1$ and $b > 2$. In case A one has strong phase separation for any density, whereas in case B one has soft phase separation at any density for $\sigma < 1$ and above a critical density

$$\rho_c = \frac{1}{b-2} \tag{4.17}$$

for $\sigma = 1$. The fluid regime has particles with density ρ_c. Hence in a finite system the macroscopic size of the condensate in the phase-separated regime is determined by the system parameter b. For an asymptotic decay faster than $2/n$ there is no condensed phase, the system is disordered for all densities.

Models with two conservation laws for which J_n decays exponentially to zero with n (case A) have been reviewed above and indeed were shown to exhibit strong phase separation at any density. In the model (4.7) with a single conservation law the current of particles out of the left domain wall, i.e., the current opposite to the bias of the individual particles, decays exponentially with domain size for $r = 0$, as demonstrated for the usual ASEP with appropriately chosen open boundary conditions [185]. The domain size dependence of the current flowing away from the right edge of a cluster (in the direction of the bias) is to leading order not a self-organized quantity, it is determined strongly by the interaction of the particle at the edge of the cluster with the surrounding particles. The interaction range is one lattice site and hence the current J_n out of the right edge of a cluster of density 1 becomes to leading order independent of the cluster size for $n > 2$. It vanishes due to the kinetic constraint $r = 0$ (zero-temperature condition). According to the criterion one expects strong phase separation, in agreement with the result reviewed above.

For $J_\infty \neq 0$ (case B) we note that in a system with two conservation laws the current inside a cluster organizes itself to a value determined by the dynamics of the reduced system with only one conservation law resulting from the absence of vacancies. This reduced system has open boundaries with inflow and outflow of particles such that the system is in the generic maximal current phase of the reduced system. It is assumed that the current flowing through a block is given by its steady-state value and is independent of its neighbouring blocks. This may be justified by the fact that the coarsening time of large domains is very long, and the domains have a chance to equilibrate long before they coarsen.

In case B one expects generically $\sigma = 1$ for the following reason. (a) In a *periodic system* the leading finite-size corrections to the current J_∞ in a canonical ensemble are given by $J_n - J_\infty = -J''_\infty \kappa/(2n)$ [178, 186]. Here J''_∞ is the curvature of the current–density relation and $\kappa = (\langle N^2\rangle - \langle N\rangle^2)/L$ is the nonequilibrium analogue of the thermodynamic compressibility which is assumed to be finite, i.e., one assumes sufficiently rapidly decaying correlations as was implied above in the derivation of the collective velocity which also requires finite compressibility. (b) There is a universal ratio c^* of the finite-size corrections to the current in the maximal current phase of a driven diffusive system (which describes the dynamics inside the growing domains) and the finite-size corrections of the canonical ensemble of a periodic system [187]. This yields leading finite-size corrections of the form (4.13) with a parameter b entirely determined by the universal constant c^* and the macroscopic

quantities J''_∞ and κ. The value of $c^* = \frac{3}{2}$ has been obtained from the exact solution of the ASEP with open boundaries [19, 20].

We stress that by definition b is a quantity that itself does not depend on system size. For systems with unknown stationary distribution, the reduced dynamics inside a cluster allows for a simple numerical measurement of b by studying the finite-size corrections of the stationary current in the reduced open system of length n. One neither needs huge lattices nor is one faced with the problem of slow relaxation of the phase separation process in the full system. Applying the criterion to the model (3.14) yields the exact value $b = \frac{3}{2}$ and hence one expects no condensation, in agreement with the exact result. For the two-lane model of Korniss *et al* [180] one obtains numerically $b \approx 0.8$ [184] and therefore one expects no condensation in contrast to the results of the Monte Carlo simulation of the full model with 10^4 lattice sites. A three-state model with KLS dynamics (2.11) inside the clusters has been shown to have $b > 2$ [178] which suggests the existence of soft phase separation in driven diffusive systems with two conservation laws.

The criterion presented above emerges from a careful analysis of the zero-range process (ZRP) which could be viewed as a generic model for domain dynamics in one dimension [184].

Depending on the rates w_n the model may or may not exhibit condensation in the thermodynamic limit, whereby the occupation number of one of the boxes becomes macroscopically large. Clearly, the rate w_n must be a decreasing function of n in order for larger blocks to be favoured and to support condensation. It is known [8, 106] that condensation occurs at any density when $w_n \to 0$ with $n \to \infty$, or when it decreases to a nonvanishing asymptotic value as b/n^σ with $\sigma < 1$; no phase separation takes place for $\sigma > 1$; for $\sigma = 1$ phase separation takes place at high densities only for $b > 2$. This model may be used to gain physical insight into the dynamics of driven one-dimensional systems. Occupied boxes represent domains of the high-density phase. The currents leaving domains are represented by the rates of the ZRP. This is done by identifying the rate w_n associated with a box containing n balls with the currents J_n leaving a domain of n particles. A bias in the currents to a certain direction may be incorporated through a bias in the ZRP dynamics. The existence of a box with a macroscopic occupation in the ZRP corresponds to phase separation in the driven model. The distribution of occupation numbers obtained from the ZRP was shown to agree with the domain size distribution of the model (3.14) [184].

It is remarkable that extending the asymptotic expansion (4.14) by a (non-universal) next-leading term c/n^2 leads to an extremely sensitive dependence of the mean domain size ξ on c [177]. The quantity ξ exhibits a sharp increase of a few orders of magnitude over a narrow range of values of c. This reflects itself in large (but finite) blocks and an apparent phase separation in direct simulations.

4.4. Coarsening

The previous discussion has focused on the stationary properties of phase separation. The dynamics below the critical density are expected to be described on the hydrodynamic scale by the mechanisms reviewed in section 3. Above the critical density a natural set-up is to start with particles uniformly distributed at the supercritical density $\rho > \rho_c$. In the beginning the excess particles condense at a few random sites. Thus there are several clusters which are essentially immobile. On the remaining sites, the distribution relaxes to its critical stationary distribution with $\rho = \rho_c$. With increasing time the larger clusters will gain particles at the expense of the smaller ones, causing the clusters to merge. Eventually, only a single

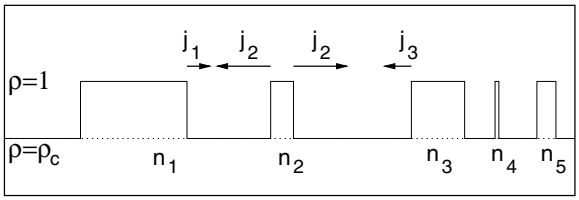

Figure 6. Coarsening of domains during phase separation. The regions between domains have relaxed to their stationary critical density. Each domain of size n_r has an outflowing current j_r determined by the dynamics inside the domain. In order to have domain growth the current should decrease with domain size, as indicated in the figure for domains 1, 2, 3. Domain 2 is expected to be eaten up by domains 1 and 3 after some time.

cluster containing all excess particles survives, which is typical for the stationary distribution (figure 6).

In case A (strong phase separation) the evolution of this state will proceed by slow diffusion against the bias, in which, for example an A-particle traverses the adjacent domain of B-particles. Using standard mean first passage time calculations the time t necessary for penetrating the complete B-domain of size n can be shown to be of order q^n where $q > 1$ is the hopping asymmetry, i.e., the ratio of hopping rates inside the B-domain. After that the A-particle travels with finite average velocity to the edge of the next A-domain. This happens after a time of order L which can be neglected compared to the penetration time. Therefore the coarsening time is of the order q^n and inverting this relation yields a logarithmic growth law for the mean domain size in the strong phase separation [172],

$$\bar{n}(t) \sim \ln t / \ln q. \tag{4.18}$$

This growth law is valid also in higher dimensions [188].

The zero-range picture may be used in order to study the coarsening dynamics of domains in the case of soft phase separation. We set $J_\infty = 1$ which only fixes an uninteresting microscopic time unit for coarsening. In [189] the coarsening dynamics are described by studying with random walk arguments the loss and gain of particles in neighbouring clusters, mediated by stationary transport in the fluid phase in between. The time that particles lost from a cluster spend in between clusters is of the same order as the time required to dissolve a cluster completely. Hence this time scale is the appropriate coarsening time scale. In the totally asymmetric case, excess particles leave a cluster site with n particles at a rate b/n and therefore the typical time required to dissolve such a cluster is given by $t_a(n) \sim n^2/b$. Inverting this scaling relation yields the growth law

$$\bar{n}(t) \sim (bt)^{1/2} \tag{4.19}$$

for the mean cluster size $\bar{n}$. From (4.19) one reads off the dynamical exponent $z = 2$ in the asymmetric condensed phase.

Because of the recurrence of a one-dimensional (1D) random walk in the case of symmetric hopping, a particle that got lost from a cluster is very likely to return to the cluster it left, in contrast to the asymmetric case. It will be trapped by the next cluster only with a probability inversely proportional to the diffusion distance, i.e., of the order $(\rho - \rho_c)/m$. So the typical time of a particle to leave a cluster is of the order $m^2/(b(\rho - \rho_c))$. Hence the coarsening time scale is given by $t_s(n) \sim t_a(\rho - \rho_c)/m \sim m^3/[b(\rho - \rho_c)]$. This results in the growth law

$$\bar{n}(t) \sim [b(\rho - \rho_c)t]^{1/3} \tag{4.20}$$

and dynamical exponent $z = 3$ in the symmetric case. A similar growth law which can be rationalized using analogous arguments is known for equilibrium Kawasaki dynamics (2.12)

at very low temperatures [190, 191]. For biased coarsening dynamics with conserved order parameter one has $z = 2$ [192, 193], in agreement with (4.19). The coarsening stops when only one macroscopic cluster is left. The typical time of a macroscopic fluctuation of the cluster size diverges exponentially with the system size L [189].

The growth laws (4.19) and (4.20) were confirmed by Monte Carlo simulations of the ZRP [189]. The critical exponents were obtained independently by numerically studying the second moment of the local density [194]. Also a universal scaling function for the cluster size distribution was obtained. At the critical density itself Monte Carlo simulations suggest dynamical exponents $z_c \approx 3$ for the asymmetric case and $z_c \approx 5$ for the symmetric case. A theoretical derivation of these exponents is lacking.

5. Conclusions and open questions

During the last decade the study of one-dimensional driven diffusive systems has contributed significantly to the understanding of critical phenomena far from equilibrium, where 'understanding' not only refers to the characterization of nonequilibrium universality classes in terms of critical exponents, but also to the identification of some of the coarse-grained dynamical mechanisms that generate these critical phenomena. Exact results for simple model systems such as the ASEP have played a crucial role in advancing and shaping this understanding. They have provided deep and detailed insights into robust generic phenomena which could then be generalized to more complicated systems.

This includes the derivation of exact hydrodynamic equations and asymptotic coarsening laws from the microscopic laws of interaction, both rigorously and using more intuitive physical arguments based on the application of conservation laws and using random walk arguments. Thus complicated collective phenomena such as shocks can be viewed as effective single-particle excitations with simple properties. Universal fluctuations determining the dynamics on intermediate scales between the microscopic lattice scale and the macroscopic Euler scale have been probed using test particles: tracer particles, second-class particles and impurity particles. Thus some similarities to the still unresolved problem of localized blockages could be established. The occurrence of stationary critical phenomena, namely spontaneous symmetry breaking, long-range order and phase separation of two types, soft and strong, could be linked to microscopic properties of the dynamics, listed in conditions (I)–(IV) of section 4. Stability criteria (3.22), (3.24) for microscopically sharp domain walls and coarsening dynamics of domains (4.15), (4.16) as well as the phase diagram of open one-species systems (1.5) can be phrased directly in terms of the macroscopic current. We conclude that the current–density relation (which is determined by microscopic parameters) is a central quantity determining the large scale physics of driven diffusive systems. This is fortunate in the sense that the current is usually relatively easy to measure or calculate. It is crucial to know the current exactly, mean-field approximations are not likely to produce even qualitatively (let alone quantitatively) correct features of the system, unless by chance certain analytic properties of the mean-field current happen to coincide with those of the exact current. However, this is not guaranteed even if bulk correlations in the stationary state are short-range.

A major open question concerns the precise relationship of the various current criteria and the microscopic criteria for phase separation. Addressing this issue leads to a series of interrelated problems, each of which is interesting in its own right.

It is not clear how to deduce the stability of a domain wall in the stationary condensed state or during the coarsening process (where the current is also stationary and independent of

the density) from the stability criteria (3.22), (3.24). This requires a more careful analysis of the behaviour of fluctuations in systems with two conservation laws and is intimately linked to a hydrodynamic description of the dynamics above the critical density, which is an open problem even within the framework of an effective ZRP description. The hydrodynamic treatment of the blockage problem in the usual ASEP [149] may provide some insight.

Moreover, it would be highly desirable to have a unified picture which allows for an application of the current criteria (4.15), (4.16) for phase separation to single-species systems satisfying one of conditions (I)–(III) and thus to predict from the dependence of the domain current on the microscopic parameters at which value of these parameters phase separation sets in. This requires a proper definition of the current out of a domain in a system with one conservation law which might then answer also the question whether soft phase separation is possible if none of conditions (I)–(III) is satisfied. In a single-species system this condensation phenomenon is reminiscent of spontaneous traffic jams in automobile traffic flow. Indeed, traffic models with nonconserved internal degrees of freedom are known to exhibit soft phase separation [195–198], but the minimal requirement for the existence of the phenomenon is not entirely clear. The role of nonconserved internal degrees of freedom in critical phenomena needs further clarification also in the theory of boundary-induced phase transitions which requires some extension [199].

In this context it would be interesting to try to predict phase separation from the properties of the current, using (3.22), (3.24). This has not been attempted yet even in systems where both the current and the existence of phase separation are known and might shed light on the possibility of phase separation between domains of nonextremal densities $\rho_i \neq 0, 1$ in translation-invariant systems. The stability criterion (3.22) rules this out for generic current–density relations in systems with one conservation law, but it is not tantamount to a no-go theorem as a current which is constant in some density range may allow for such phase separation.

A further promising and closely related direction of research concerns the hydrodynamics of systems with more than one conservation law. The lack of a full hydrodynamic description of particle systems with two conservation law still constitutes a major gap in understanding both dynamic and critical stationary critical phenomena. The selection of the physical solution using a regularization by adding a phenomenological viscosity term is not fully understood, as the nature of such a term might not be as arbitrary as for systems with one conservation law. This may be of importance not only for the bulk critical phenomena reviewed here, but also for boundary-induced spontaneous symmetry breaking [200] and steady-state selection [31] in open systems. There is no theory of boundary-induced phase transitions with an extremal principle analogous to (1.5) that could explain quantitative features of the phase diagram in terms of effective boundary densities as is possible for single-species models. Hence it is difficult to make predictions of the stationary behaviour in open systems. For systems with more than two conservation laws the problem is likely to be even more intricate, but possibly also even more fascinating.

Universal properties of fluctuations which manifest themselves on scales below the Euler scale are not readily accessible with the analytical methods reviewed in this paper. However, with the tools of Bethe ansatz and random matrix theory they have become amenable to exact treatment in the framework of the ASEP [5, 201]. It is natural to try to apply these techniques to the general two-species ASEP or at least to the integrable cases. Universal quantities—critical exponents and scaling functions, but also universal dynamical mechanisms such as evolution of shocks and coarsening—derived from studying driven diffusive systems are ultimately expected to be useful also in the investigation of real systems such as those listed above. Therefore one needs to understand the role of the lattice in the phenomena discussed above.

Passing to driven diffusive systems defined in spatial continuum by taking appropriate limits may give an answer.

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